

# Modular Quantizations of Lie Algebras of Cartan Type $K$ via Drinfeld Twists of Jordanian Type

Zhaojia Tong and Naihong Hu\*

**ABSTRACT.** We construct explicit Drinfel'd twists of Jordanian type for the generalized Cartan type  $K$  Lie algebras in characteristic 0 and obtain the corresponding quantizations, especially their integral forms. By making modular reductions including modulo  $p$  and modulo  $p$ -restrictedness reduction, and base changes, we derive certain modular quantizations of the restricted universal enveloping algebra  $u(\mathbf{K}(2n+1; \underline{1}))$  for the restricted simple Lie algebra of Cartan type  $K$  in characteristic  $p$ . They are new Hopf algebras of noncommutative and noncocommutative and with dimension  $p^{p^{2n+1}+1}$  (if  $2n+4 \not\equiv 0 \pmod{p}$ ) or  $p^{p^{2n+1}}$  (if  $2n+4 \equiv 0 \pmod{p}$ ) over a truncated  $p$ -polynomials ring, which also contain the well-known Radford algebras as Hopf subalgebras. Some open questions are proposed.

This paper is a continuation of [14, 15, 28] in which quantizations of Cartan type Lie algebras of types  $W$ ,  $S$  and  $H$  were studied. In the present paper, we continue to treat the same questions both for the generalized Cartan type  $K$  Lie algebras in characteristic 0 (for the definition, see [24]) and for the restricted simple contact algebra  $\mathbf{K}(2n+1; \underline{1})$  of Cartan type  $K$  in the modular case (for the definition, see [25], [26]). This work, together with papers [14, 15, 28], completes the work of this kinds of modular quantizations of Jordanian type for the Cartan type Lie algebras over a field  $\mathcal{K}$  with  $\text{char}(\mathcal{K}) \geq 7$ .

We survey some previous related work. After the works on quantum groups which were introduced by Drinfel'd [4] and Jimbo [16], Drinfeld in [5] raised the question of the existence of a universal quantization for Lie bialgebras. Etingof-Kazhdan gave a positive answer to this question in [7, 8], where the Lie bialgebras they considered including finite- and infinite-dimensional ones are those defined by generalized Cartan matrices. Enriquez-Halbout showed that any coboundary Lie bialgebra, in principle, can be quantized via a certain Etingof-Kazhdan quantization functor [6], and Geer [10] further extended Etingof-Kazhdan's work from Lie bialgebras to the setting of Lie superbialgebras. After the work [7, 8], it is natural to consider the quantizations of Lie algebras of Cartan type that are

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defined by differential operators. Grunspan [13] obtained the quantization of the (infinite-dimensional) Witt algebra  $\mathbf{W}$  in characteristic 0 using the twist due to Giaquinto-Zhang [11], however, his approach didn't work for the quantum version of its simple modular Witt algebra  $\mathbf{W}(1; \underline{1})$  in characteristic  $p$ . Hu-Wang in [14] obtained the quantizations of the generalized Witt algebra  $\mathbf{W}$  in characteristic 0 and the Jacobson-Witt algebra  $\mathbf{W}(n; \underline{1})$  in characteristic  $p$ ; these are new families of noncommutative and noncocommutative Hopf algebras of dimension  $p^{1+np^n}$  in characteristic  $p > 0$ , while in rank 1 case, the work not only recovered Grunspan's one in characteristic 0, but also gave the correct modular quantum version. Note that the concept of "modular" quantization and relevant methods were settled in the work [14] and [15] (also see [28]).

Although, in principle, the possibility to quantize an arbitrary Lie bialgebra has been proved ([7, 8, 6, 9, 10], etc.), an explicit formulation of Hopf operations remains nontrivial. In particular, only a few kinds of twists were known with explicit expressions, see [11, 17, 18, 21, 23] and the references therein. In this research, we start with an explicit Drinfel'd twist due to [11, 13] and in fact, this Drinfel'd twist is essentially a variation (see the proof in [15]) of the Jordanian twist which first appeared, using different expression, in Coll-Gerstenhaber-Giaquinto's paper [3], and recently used extensively by Kulish et al (see [17, 18], etc.). Using this explicit Drinfel'd twist we obtain *vertical basic* twists and *horizontal basic* twists for the generalized Cartan type  $K$  Lie algebras and the corresponding quantizations in characteristic 0. These basic twists can afford many more Drinfel'd twists, likewise on types  $W$ ,  $S$ ,  $H$ . To study the modular case, what we discuss first involves the arithmetic property of quantizations, for working out their quantization integral forms. To this end, we have to work over the so-called "*positive*" part subalgebra  $\mathbf{K}^+$  of the generalized Cartan type  $K$  Lie algebra. This is the crucial observation here. It is an infinite-dimensional simple Lie algebra defined over a field of characteristic 0, while, over a field of characteristic  $p$ , it contains a maximal ideal  $J_1$  and the corresponding quotient is exactly the algebra  $\mathbf{K}'(2n+1; \underline{1})$ . Its derived subalgebra  $\mathbf{K}(2n+1; \underline{1}) = \mathbf{K}'(2n+1; \underline{1})^{(1)}$  is a Cartan type restricted simple modular Lie algebra of  $K$  type. Secondly, in order to yield the *expected* finite-dimensional quantizations of the restricted universal enveloping algebra of the Contact algebra  $\mathbf{K}(2n+1; \underline{1})$ , we need to carry out the modular reduction process: *modulo  $p$  reduction* and *modulo " $p$ -restrictedness" reduction*, during which we have to take the suitable *base changes*. These are the other two crucial technical points. Our work gives a new class of noncommutative and noncocommutative Hopf algebras of prime-power dimension in characteristic  $p$ , which is significant to recognize the Kaplansky's problem.

The paper is organized as follows. In Section 1, we recall some definitions and basic facts related to the Cartan type  $K$  Lie algebra and Drinfel'd twist. In Section 2, we construct the Drinfel'd twists for the generalized Cartan type  $K$  Lie algebra, including *vertical basic* twists and *horizontal basic* twists. In Section 3, we quantize explicitly Lie bialgebra structures of the generalized Cartan type  $K$  Lie algebra by the *vertical basic* Drinfel'd twists, and using the similar methods as in type  $H$ , we obtain the quantizations of the

restricted universal enveloping algebra of the Contact algebra  $\mathbf{K}(2n+1; \underline{1})$ . In Section 4, using the *horizontal* twists, we get some new modular quantizations of horizontal type of  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$ . In Section 5, we give another two kinds of Drinfel'd twists and get some new modular quantizations of horizontal type of  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$ . Finally, we present some open questions.

## 1. Preliminaries

**1.1. The generalized Cartan type  $K$  Lie algebra and its subalgebra.** We recall the definitions of the generalized Cartan type  $K$  Lie algebras and the restricted simple Lie algebras of Cartan type from [24, 26] and some basics about their structures.

Let  $\mathbb{F}$  be a field of characteristic 0.  $\mathbb{Q}_{2n+1} = \mathbb{F}[x_{-n}^{\pm 1}, \dots, x_{-1}^{\pm 1}, x_0^{\pm 1}, x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  ( $n \in \mathbb{N}$ ) be Laurent polynomial algebra, and  $\partial_i$  coincide with the degree operator  $x_i \frac{\partial}{\partial x_i}$ . Set  $T = \bigoplus_{i=1}^n \mathbb{Z} \partial_{-i} \oplus \mathbb{Z} \partial_i \oplus \mathbb{Z} \partial_0$ ,  $x^\alpha = x_{-n}^{\alpha_{-n}} \cdots x_{-1}^{\alpha_{-1}} x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ , for  $\alpha = (\alpha_{-n}, \dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots, \alpha_n)$ . Define  $\mathbf{W} = \text{Der}(\mathbb{Q}_{2n+1}) = \text{Span}_{\mathbb{F}}\{x^\alpha \partial \mid \alpha \in \mathbb{Z}^{2n+1}, \partial \in T\}$ . Then  $\mathbf{W}$  is a Lie algebra of generalized Witt type under the bracket

$$[x^\alpha \partial, x^\beta \partial'] = x^{\alpha+\beta} (\partial(\beta) \partial' - \partial'(\alpha) \partial), \quad \forall \alpha, \beta \in \mathbb{Z}^{2n+1}, \partial, \partial' \in T,$$

where  $\partial(\beta) = \langle \partial, \beta \rangle = \langle \beta, \partial \rangle = \sum_{i=1}^n a_{-i} \beta_{-i} + a_i \beta_i + a_0 \beta_0$ , for  $\partial = \sum_{i=1}^n a_{-i} \partial_{-i} + a_i \partial_i + a_0 \partial_0$  and  $\beta = (\beta_{-n}, \dots, \beta_{-1}, \beta_0, \beta_1, \dots, \beta_n)$ . The bilinear map  $\langle \cdot, \cdot \rangle : T \times \mathbb{Z}^{2n+1} \longrightarrow \mathbb{Z}$  is nondegenerate in the sense that

$$\partial(\alpha) = \langle \partial, \alpha \rangle = 0 \quad (\forall \partial \in T), \implies \alpha = \underline{0},$$

$$\partial(\alpha) = \langle \partial, \alpha \rangle = 0 \quad (\forall \alpha \in \mathbb{Z}^{2n+1}), \implies \partial = 0.$$

where  $\underline{0} = (0, \dots, 0)$ .  $\mathbf{W}$  is an infinite dimensional Lie algebra over  $\mathbb{F}$ .

Consider the linear map  $\mathcal{D}_K : \mathbb{Q}_{2n+1} \longrightarrow \mathbf{W}$  defined by

$$\begin{aligned} \mathcal{D}_K(x^\alpha) &= \left(2 - \sum_{i=1}^n (\alpha_i + \alpha_{-i})\right) x^\alpha \frac{\partial}{\partial x_0} \\ &\quad + \sum_{i=1}^n \left( (\alpha_0 x^{\alpha+\epsilon_i-\epsilon_0} + \alpha_{-i} x^{\alpha-\epsilon_{-i}}) \frac{\partial}{\partial x_i} + (\alpha_0 x^{\alpha+\epsilon_{-i}-\epsilon_0} - \alpha_i x^{\alpha-\epsilon_i}) \frac{\partial}{\partial x_{-i}} \right). \end{aligned}$$

So we get

$$\mathcal{D}_K(x^\alpha) = \left(2 - \sum_{i=1}^n (\alpha_i + \alpha_{-i})\right) x^{\alpha-\epsilon_0} \partial_0 + \sum_{i=1}^n \alpha_0 x^{\alpha-\epsilon_0} (\partial_i + \partial_{-i}) + \sum_{i=1}^n x^{\alpha-\epsilon_i-\epsilon_{-i}} (\alpha_{-i} \partial_i - \alpha_i \partial_{-i}).$$

It is easy to see that  $\mathcal{D}_K$  is injective. Then for any  $x^\alpha, x^\beta \in \mathbb{Q}_{2n+1}$ , the Lie bracket becomes

$$\begin{aligned} [\mathcal{D}_K(x^\alpha), \mathcal{D}_K(x^\beta)] &= \mathcal{D}_K \left[ \left(2x^\alpha - \sum_{i=1}^n (\alpha_i + \alpha_{-i}) x^\alpha\right) \beta_0 x^{\beta-\epsilon_0} - \left(2x^\beta - \sum_{i=1}^n (\beta_i + \beta_{-i}) x^\beta\right) \alpha_0 x^{\alpha-\epsilon_0} \right. \\ &\quad \left. + \sum_{i=1}^n (\alpha_{-i} \beta_i - \alpha_i \beta_{-i}) x^{\alpha+\beta-\epsilon_i-\epsilon_{-i}} \right] \end{aligned}$$

$$= \mathcal{D}_K \left( \left( (2 - \sum_{i=1}^n (\alpha_i + \alpha_{-i})) \beta_0 - (2 - \sum_{i=1}^n (\beta_i + \beta_{-i})) \alpha_0 \right) x^{\alpha + \beta - \epsilon_0} + \sum_{i=1}^n (\alpha_{-i} \beta_i - \alpha_i \beta_{-i}) x^{\alpha + \beta - \epsilon_i - \epsilon_{-i}} \right).$$

It follows that  $\mathbf{K} = \mathcal{D}_K(\mathbb{F}[x_{-n}^{\pm 1}, \dots, x_{-1}^{\pm 1}, x_0^{\pm 1}, x_1^{\pm 1}, \dots, x_n^{\pm 1}])$ , and  $\mathbf{K}^+ = \mathcal{D}_K(\mathbb{F}[x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n])$  are Lie subalgebras of  $\mathbf{W}$ .  $\mathbf{K}, \mathbf{K}^+$  are simple algebras with basis  $\mathcal{D}_K(x^\alpha)$ ,  $\alpha \in \mathbb{Z}^{2n+1}$ ,  $\mathcal{D}_K(x^\alpha)$ ,  $\alpha \in \mathbb{Z}_+^{2n+1}$ , respectively.  $\mathbf{K}$  is the Lie algebra of generalized Cartan type  $K$  ([24]).

**1.2. The Contact algebra  $\mathbf{K}(2n+1; \underline{1})$ .** Assume that  $\text{char}(\mathcal{K}) = p$ , then by definition (see [25]), the Jacobson-Witt algebra  $\mathbf{W}(2n+1; \underline{1})$  is a restricted simple Lie algebra over a field  $\mathcal{K}$ . Its structure of  $p$ -Lie algebra is given by  $D^{[p]} = D^p$ ,  $\forall D \in \mathbf{W}(2n+1; \underline{1})$  with a basis  $\{x^{(\alpha)} D_j \mid -n \leq j \leq n, 0 \leq \alpha \leq \tau\}$ , where  $\tau = (p-1, \dots, p-1) \in \mathbb{N}^{2n+1}$ ;  $\epsilon_i = (\delta_{-n,i}, \dots, \delta_{-1,i}, \delta_{0,i}, \delta_{1,i}, \dots, \delta_{n,i})$  with  $x^{(\epsilon_i)} = x_i$ ,  $x^{(\alpha)} \in \mathcal{O}(2n+1; \underline{1}) = \text{Span}_{\mathcal{K}}\{x^{(\alpha)} \mid 0 \leq \alpha \leq \tau\}$ , the restricted divided power algebra with  $x^{(\alpha)} x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{\alpha+\beta}$  and a convention:  $x^{(\alpha)} = 0$  if  $\alpha$  has a component  $\alpha_j < 0$  or  $\geq p$ , where  $\binom{\alpha+\beta}{\alpha} = \prod_{i=1}^n \binom{\alpha_i+\beta_i}{\alpha_i} \binom{\alpha_{-i}+\beta_{-i}}{\alpha_{-i}} \binom{\alpha_0+\beta_0}{\alpha_0}$ . Define  $\mathcal{D}_K : \mathcal{O}(2n+1; \underline{1}) \rightarrow \mathbf{W}(2n+1; \underline{1})$ , where  $\mathcal{D}_K(x^{(\alpha)}) = (2 - \sum_{i=1}^n (\alpha_i + \alpha_{-i})) x^{(\alpha)} D_0 + \sum_{i=1}^n x^{(\alpha - \epsilon_0)} (x^{(\epsilon_i)} D_i + x^{(\epsilon_{-i})} D_{-i}) + \sum_{i=1}^n (x^{(\alpha - \epsilon_{-i})} D_i - x^{(\alpha - \epsilon_i)} D_{-i})$ . Then the subspace  $\mathbf{K}'(2n+1; \underline{1}) = \mathcal{D}_K(\mathcal{O}(2n+1; \underline{1}))$  is a Lie subalgebra of  $\mathbf{W}(2n+1; \underline{1})$ . Its derived algebra  $\mathbf{K}(2n+1; \underline{1})$  is called the Contact algebra. Define  $\|\alpha\| = |\alpha| + \alpha_0 - 2$ , Then  $\mathbf{K}(2n+1; \underline{1}) = \bigoplus_{i=-2}^s \mathbf{K}(2n+1; \underline{1})_i$  is graded, where  $\mathbf{K}(2n+1; \underline{1})_i = \text{Span}_{\mathcal{K}}\{\mathcal{D}_K(x^{(\alpha)}) \mid 0 \leq \alpha \leq \tau, \|\alpha\| = i\}$ ,  $s = (2n+2)(p-1)-2$  when  $2n+4 \not\equiv 0 \pmod{p}$ ;  $\mathbf{K}(2n+1; \underline{1})_i = \text{Span}_{\mathcal{K}}\{\mathcal{D}_K(x^{(\alpha)}) \mid 0 \leq \alpha < \tau, \|\alpha\| = i\}$ ,  $s = (2n+2)(p-1)-3$  when  $2n+4 \equiv 0 \pmod{p}$ . Then by Theorem 5.5 of [25], we have

$$\mathbf{K}(2n+1; \underline{1}) = \begin{cases} \text{Span}_{\mathcal{K}}\{\mathcal{D}_K(x^{(\alpha)}) \mid x^{(\alpha)} \in \mathcal{O}(2n+1; \underline{1}), 0 \leq \alpha \leq \tau\}, & \text{if } 2n+4 \not\equiv 0 \pmod{p}, \\ \text{Span}_{\mathcal{K}}\{\mathcal{D}_K(x^{(\alpha)}) \mid x^{(\alpha)} \in \mathcal{O}(2n+1; \underline{1}), 0 \leq \alpha < \tau\}, & \text{if } 2n+4 \equiv 0 \pmod{p}, \end{cases}$$

is a Lie  $p$ -subalgebra of  $\mathbf{W}(2n+1; \underline{1})$  with restricted gradation.

By definition (cf. [26]), the restricted universal enveloping algebra  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$  is isomorphic to  $U(\mathbf{K}(2n+1; \underline{1}))/I$ , where  $I$  is the Hopf ideal of  $U(\mathbf{K}(2n+1; \underline{1}))$  generated by  $(\mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}}))^p - \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$ ,  $(\mathcal{D}_K(x^{\epsilon_0}))^p - \mathcal{D}_K(x^{\epsilon_0})$ ,  $(\mathcal{D}_K(x^{(\alpha)}))^p$  with  $\alpha \neq \epsilon_k + \epsilon_{-k}$ ,  $\epsilon_0$ ,  $1 \leq k \leq n$ . Since

$$\dim_{\mathcal{K}}((\mathbf{K}(2n+1; \underline{1}))) = \begin{cases} p^{2n+1}, & \text{if } 2n+4 \not\equiv 0 \pmod{p}, \\ p^{2n+1} - 1, & \text{if } 2n+4 \equiv 0 \pmod{p}, \end{cases}$$

we have

$$\dim_{\mathcal{K}}(\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))) = \begin{cases} p^{p^{2n+1}}, & \text{if } 2n+4 \not\equiv 0 \pmod{p}, \\ p^{p^{2n+1}-1}, & \text{if } 2n+4 \equiv 0 \pmod{p}. \end{cases}$$

**1.3. Quantization by Drinfel'd twists.** The following result is well-known ([4]).

LEMMA 1.1. *Let  $(A, m, \iota, \Delta_0, \varepsilon, S_0)$  be a Hopf algebra over a commutative ring. A Drinfel'd twist  $\mathcal{F}$  on  $A$  is an invertible element of  $A \otimes A$  such that*

$$\begin{aligned} (\mathcal{F} \otimes 1)(\Delta_0 \otimes \text{Id})(\mathcal{F}) &= (1 \otimes \mathcal{F})(\text{Id} \otimes \Delta_0)(\mathcal{F}), \\ (\varepsilon \otimes \text{Id})(\mathcal{F}) &= 1 = (\text{Id} \otimes \varepsilon)(\mathcal{F}). \end{aligned}$$

*Then,  $w = m(\text{Id} \otimes S_0)(\mathcal{F})$  is invertible in  $A$  with  $w^{-1} = m(S_0 \otimes \text{Id})(\mathcal{F}^{-1})$ .*

*Moreover, if we define  $\Delta : A \longrightarrow A \otimes A$  and  $S : A \longrightarrow A$  by*

$$\Delta(a) = \mathcal{F} \Delta_0 \mathcal{F}^{-1}, \quad S(a) = w S_0(a) w^{-1},$$

*then  $(A, m, \iota, \Delta, \varepsilon, S)$  is a new Hopf algebra, called the twisting of  $A$  by Drinfel'd twist  $\mathcal{F}$ .*

Let  $\mathbb{F}[[t]]$  be a ring of formal power series over a field  $\mathbb{F}$  with  $\text{char}(\mathbb{F}) = 0$ . Assume that  $L$  is a triangular Lie algebra over  $\mathbb{F}$  with a classical Yang-Baxter  $r$ -matrix  $r$ . (see [4], [8]). Let  $U(L)$  denote the universal enveloping algebra of  $L$ , with the standard Hopf algebra  $(U(L), m, \iota, \Delta_0, \varepsilon_0, S_0)$ .

Let us consider the topologically free  $\mathbb{F}[[t]]$ -algebra  $U(L)[[t]]$  (for definition, see [8], p.4), which can be viewed as an associative  $\mathbb{F}$ -algebra of formal power series with coefficients in  $U(L)$ . Naturally,  $U(L)[[t]]$  equips with an induced Hopf algebra structure arising from that on  $U(L)$ . By abuse of notation, we denote it by  $(U(L)[[t]], m, \iota, \Delta_0, \varepsilon_0, S_0)$ .

DEFINITION 1.2. [14] For a triangular Lie bialgebra  $L$  over  $\mathbb{F}$  with  $\text{char}(\mathbb{F}) = 0$ ,  $U(L)[[t]]$  is called a *quantization* of  $U(L)$  by a Drinfel'd twist  $\mathcal{F}$  over  $U(L)[[t]]$  if  $U(L)[[t]]/tU(L)[[t]] \cong U(L)$ , and  $\mathcal{F}$  is determined by its  $r$ -matrix  $r$  (namely, its Lie bialgebra structure).

## 2. Drinfel'd twist in $U(\mathbf{K})[[t]]$

**2.1. Construction of Drinfel'd twists.** Let  $L$  be a Lie algebra containing linearly independent elements  $h$  and  $e$  satisfying  $[h, e] = e$ ; then the classical Yang-Baxter  $r$ -matrix  $r = h \otimes e - e \otimes h$  equips  $L$  with the structure of a triangular coboundary Lie bialgebra (see [19]). To describe a quantization of  $U(L)$  by a Drinfel'd twist  $\mathcal{F}$  over  $U(L)[[t]]$ , we need an explicit construction for such a Drinfel'd twist. In what follows, we shall see that such a Drinfel'd twist depends on the choice of two distinguished elements  $h$  and  $e$  arising from its  $r$ -matrix  $r$ .

For any element of a unital  $R$ -algebra ( $R$  a ring) and  $a \in R$ , we set

$$\begin{aligned} x_a^{(m)} &:= (x+a)(x+a+1) \cdots (x+a+m-1), \\ x_a^{[m]} &:= (x+a)(x+a-1) \cdots (x+a-m+1), \end{aligned}$$

and then denote  $x^{(m)} := x_0^{(m)}$ ,  $x^{[m]} := x_0^{[m]}$ .

Note that  $h$  and  $e$  satisfy the following equalities

$$(2.1) \quad e^s \cdot h_a^{[m]} = h_{a-s}^{[m]} \cdot e^s,$$

$$(2.2) \quad e^s \cdot h_a^{(m)} = h_{a-s}^{(m)} \cdot e^s,$$

where  $m, s$  are non-negative integers,  $a \in \mathbb{F}$ .

For  $a \in \mathbb{F}$ , we set  $\mathcal{F}_a = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_a^{[r]} \otimes e^r t^r$ ,  $F_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes e^r t^r$ ,  $u_a = m \cdot (S_0 \otimes \text{Id})(F_a)$ ,  $v_a = m \cdot (\text{Id} \otimes S_0)(\mathcal{F}_a)$ . Write  $\mathcal{F} = \mathcal{F}_0$ ,  $F = F_0$ ,  $u = u_0$ ,  $v = v_0$ . Since  $S_0(h_a^{(r)}) = (-1)^r h_{-a}^{[r]}$  and  $S_0(e^r) = (-1)^r e^r$ , one has  $v_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{[r]} e^r t^r$ ,  $u_b = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_{-b}^{[r]} e^r t^r$ .

LEMMA 2.1. ([13]) For  $a, b \in \mathbb{F}$ , one has

$$\mathcal{F}_a F_b = 1 \otimes (1 - et)^{a-b}, \quad \text{and} \quad v_a u_b = (1 - et)^{-(a+b)}.$$

COROLLARY 2.2. For  $a \in \mathbb{F}$ ,  $\mathcal{F}_a$  and  $u_a$  are invertible with  $\mathcal{F}_a^{-1} = F_a$  and  $u_a^{-1} = v_{-a}$ . In particular,  $\mathcal{F}^{-1} = F$  and  $u^{-1} = v$ .

LEMMA 2.3. ([14]) For any positive integers  $r$ , we have

$$\Delta_0(h^{[r]}) = \sum_{i=0}^r \binom{r}{i} h^{[i]} \otimes h^{[r-i]}.$$

Furthermore,  $\Delta_0(h^{[r]}) = \sum_{i=0}^r \binom{r}{i} h_{-s}^{[i]} \otimes h_s^{[r-i]}$  for any  $s \in \mathbb{F}$ .

PROPOSITION 2.4. ([13, 14]) If a Lie algebra  $L$  contains a two-dimensional solvable Lie subalgebra with a basis  $\{h, e\}$  satisfying  $[h, e] = e$ , then  $\mathcal{F} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]} \otimes e^r t^r$  is a Drinfel'd twist on  $U(L)[[t]]$ .

REMARK 2.5. ([15]) Kulish et al used early the so-called *Jordanian twist* (see [17]) with the two-dimensional carrier subalgebra  $B(2)$  such that  $[H, E] = E$ , defined by the canonical twisting element

$$\mathcal{F}_J^c = \exp(H \otimes \sigma(t)), \quad \sigma(t) = \ln(1 + Et),$$

where  $\exp(X) = \sum_{i=0}^{\infty} \frac{X^i}{i!}$  and  $\ln(1 + X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} X^n$ .

By the proof of [15], we can rewrite the twist  $\mathcal{F}$  in Proposition 2.4 as

$$\mathcal{F} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} H^{[r]} \otimes E^r t^r = \exp(H \otimes \sigma'(t)), \quad \sigma'(t) = \ln(1 - Et),$$

where  $[H, -E] = -E$ . So there is no difference between the twists  $\mathcal{F}$  and  $\mathcal{F}_J^c$ . They are essentially the same up to an isomorphism on the carrier subalgebra  $B(2)$ .

**2.2. Basic Drinfel'd twists.** Take two distinguished elements  $h = \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$ ,  $e = \mathcal{D}_K(x^\alpha)$ , such that  $[h, e] = e$ , where  $1 \leq k \leq n$ . It is easy to see that  $\alpha_k - \alpha_{-k} = 1$ . Using the result of [19], we have the following:

PROPOSITION 2.6. There is a triangular Lie bialgebra structure on  $\mathbf{K}$ , given by the classical Yang-Baxter  $r$ -matrix

$$r := \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}}) \otimes \mathcal{D}_K(x^\alpha) - \mathcal{D}_K(x^\alpha) \otimes \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}}), \quad 1 \leq k \leq n,$$

where  $\alpha \in \mathbb{Z}^{2n+1}$ ,  $\alpha_k - \alpha_{-k} = 1$ , and  $[\mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}}), \mathcal{D}_K(x^\alpha)] = \mathcal{D}_K(x^\alpha)$ .  $\square$

Fix two distinguished elements  $h = \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$ ,  $e = \mathcal{D}_K(x^\alpha)$ , with  $\alpha_k - \alpha_{-k} = 1$ , then  $\mathcal{F} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]} \otimes e^r t^r$  is a Drinfel'd twist on  $U(\mathbf{K})[[t]]$ . But the coefficients of the quantizations of standard Hopf structure  $(U(\mathbf{K})[[t]], m, \iota, \Delta_0, S_0, \varepsilon_0)$  by  $\mathcal{F}$  may be not integral. In order to get integral forms, it suffices to consider what conditions are needed for those coefficients to be integers.

LEMMA 2.7. ([13]) *For any  $a, k, \ell \in \mathbb{Z}$ ,  $a^\ell \prod_{j=0}^{\ell-1} (k+ja)/\ell!$  is an integer.*  $\square$

From this Lemma, we are interested in the following two simple cases:

- (i)  $h = \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$ ,  $e = \mathcal{D}_K(x^{2\epsilon_k + \epsilon_{-k}})$  ( $1 \leq k \leq n$ );
- (ii)  $h = \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$ ,  $e = \mathcal{D}_K(x^{\epsilon_k + \epsilon_m})$  ( $1 \leq k \neq |m| \leq n$ ).

Let  $\mathcal{F}(k)$  be the corresponding Drinfel'd twist in case (i) and  $\mathcal{F}(k; m)$  the corresponding Drinfel'd twist in case (ii).

DEFINITION 2.8.  $\mathcal{F}(k)$  ( $1 \leq k \leq n$ ) are called *vertical basic Drinfel'd twists*;  $\mathcal{F}(k; m)$  ( $1 \leq k \neq |m| \leq n$ ) are called *horizontal basic Drinfel'd twists*.

REMARK 2.9. In case (i): we get  $n$  vertical basic Drinfel'd twists  $\mathcal{F}(1), \dots, \mathcal{F}(n)$ , over  $U(\mathbf{K}_{\mathbb{Z}}^+) [[t]]$ . It is interesting to consider the products of some basic Drinfel'd twists, one can get many more new Drinfel'd twists which will lead to many more new complicated quantizations not only over the  $U(\mathbf{K}_{\mathbb{Z}}^+) [[t]]$ , but over the  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$  as well, via our modulo reduction approach developed in the next section.

In case (ii): according to the parametrization of twists  $\mathcal{F}(k; m)$ , we obtain  $2n(n-1)$  horizontal basic Drinfel'd twists over  $U(\mathbf{K}_{\mathbb{Z}}^+) [[t]]$ . We will discuss these twists and corresponding quantizations in Section 4.

**2.3. More Drinfel'd twists.** We consider the products of pairwise different and mutually commutative basic Drinfel'd twists and can get many more new complicated quantizations not only over the  $U(\mathbf{K}_{\mathbb{Z}}^+) [[t]]$ , but over the  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$  as well. Note that  $[\mathcal{F}(k), \mathcal{F}(k')] = 0$  for  $1 \leq k \neq k' \leq n$ . This fact, according to the definition of  $\mathcal{F}(k)$ , implies the commutative relations in the case when  $1 \leq k \neq k' \leq n$ :

$$(2.3) \quad \begin{aligned} (\mathcal{F}(k) \otimes 1)(\Delta_0 \otimes \text{Id})(\mathcal{F}(k')) &= (\Delta_0 \otimes \text{Id})(\mathcal{F}(k))(\mathcal{F}(k') \otimes 1), \\ (1 \otimes \mathcal{F}(k))(\text{Id} \otimes \Delta_0)(\mathcal{F}(k')) &= (\text{Id} \otimes \Delta_0)(\mathcal{F}(k))(1 \otimes \mathcal{F}(k')), \end{aligned}$$

which give rise to the following property.

THEOREM 2.10.  $\mathcal{F}(k)\mathcal{F}(k')$  ( $1 \leq k \neq k' \leq n$ ) is still a Drinfel'd twist on  $U(\mathbf{K}_{\mathbb{Z}}^+) [[t]]$ .

PROOF. Similar to the proof of Theorem 2.9 in [28].  $\square$

More generally, we have the following

COROLLARY 2.11. Let  $\mathcal{F}(k_1), \dots, \mathcal{F}(k_m)$  be  $m$  pairwise different basic Drinfel'd twists and  $[\mathcal{F}(k_i), \mathcal{F}(k_s)] = 0$  for all  $1 \leq i \neq s \leq m$ . Then  $\mathcal{F}(k_1) \cdots \mathcal{F}(k_m)$  is still a Drinfel'd twist.

We denote  $\mathcal{F}_m = \mathcal{F}(k_1) \cdots \mathcal{F}(k_m)$  and its length as  $m$ . We shall show that the twisted structures given by Drinfel'd twists with different product-length are nonisomorphic.

**DEFINITION 2.12.** ([12], [15]) A Drinfel'd twist  $\mathcal{F} \in A \otimes A$  on any Hopf algebra  $A$  is called *compatible* if  $\mathcal{F}$  commutes with the coproduct  $\Delta_0$ .

In other words, twisting a Hopf algebra  $A$  with a *compatible* twist  $\mathcal{F}$  gives exactly the same Hopf structure, that is,  $\Delta_{\mathcal{F}} = \Delta_0$ . The set of *compatible* twists on  $A$  thus forms a group.

**LEMMA 2.13.** ([12]) Let  $\mathcal{F} \in A \otimes A$  be a Drinfel'd twist on a Hopf algebra  $A$ . Then the twisted structure induced by  $\mathcal{F}$  coincides with the structure on  $A$  if and only if  $\mathcal{F}$  is a compatible twist.

Using the same proof as in Theorem 2.10, we obtain

**LEMMA 2.14.** ([15]) Let  $\mathcal{F}, \mathcal{G} \in A \otimes A$  be Drinfel'd twists on a Hopf algebra  $A$  with  $\mathcal{F}\mathcal{G} = \mathcal{G}\mathcal{F}$  and  $\mathcal{F} \neq \mathcal{G}$ . Then  $\mathcal{F}\mathcal{G}$  is a Drinfel'd twist. Furthermore,  $\mathcal{G}$  is a Drinfel'd twist on  $A_{\mathcal{F}}$ ,  $\mathcal{F}$  is a Drinfel'd twist on  $A_{\mathcal{G}}$  and  $\Delta_{\mathcal{F}\mathcal{G}} = (\Delta_{\mathcal{F}})_{\mathcal{G}} = (\Delta_{\mathcal{G}})_{\mathcal{F}}$ .

**PROPOSITION 2.15.** Drinfel'd twists  $\mathcal{F}^{\zeta(i)} = \mathcal{F}(1)^{\zeta_1} \cdots \mathcal{F}(n)^{\zeta_n}$  (where  $\zeta(i) = (\zeta_1, \dots, \zeta_n) = \underbrace{(1, \dots, 1, 0, \dots, 0)}_i \in \mathbb{Z}_2^n$ ) lead to  $n$  different twisted Hopf algebra structures on  $U(\mathbf{K}_{\mathbb{Z}}^+)[[t]]$ .

### 3. Quantizations of vertical type for Lie bialgebra of Cartan type $K$

In this subsection, we explicitly quantize the Lie bialgebra  $\mathbf{K}$  by the vertical basic Drinfel'd twists, and obtain certain quantizations of the restricted universal enveloping algebra  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$  by the modular reduction and base changes.

**3.1. Quantization integral forms of the  $\mathbb{Z}$ -form  $\mathbf{K}_{\mathbb{Z}}^+$  in characteristic 0.** For the universal enveloping algebra  $U(\mathbf{K})$  for the Lie algebra  $\mathbf{K}$  over  $\mathbb{F}$ , denote by  $(U(\mathbf{K}), m, \iota, \Delta_0, S_0, \varepsilon_0)$  the standard Hopf algebra structure. We can perform the process of twisting the standard Hopf structure by the vertical Drinfel'd twist  $\mathcal{F}$ . We need to give some commutative relations, which are important to the quantizations of Lie bialgebra structure of  $\mathbf{K}$  in the sequel.

**LEMMA 3.1.** For  $h = \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$  in  $\mathbf{K}$ ,  $a \in \mathbb{F}$ ,  $m$  a non-negative integer, the following equalities hold in  $U(\mathbf{K})$ :

$$(3.1) \quad \mathcal{D}_K(x^\alpha) \cdot h_a^{[m]} = h_{a+(\alpha_k - \alpha_{-k})}^{[m]} \cdot \mathcal{D}_K(x^\alpha),$$

$$(3.2) \quad \mathcal{D}_K(x^\alpha) \cdot h_a^{\langle m \rangle} = h_{a+(\alpha_k - \alpha_{-k})}^{\langle m \rangle} \cdot \mathcal{D}_K(x^\alpha).$$

To simplify the formulas, let us introduce the operator  $d^{(\ell)}$  on  $U(\mathbf{K})$  defined by  $d^{(\ell)} = \frac{1}{\ell!}(\text{ad } e)^\ell$ . So we can get



LEMMA 3.2. For  $\mathcal{D}_K(x^\alpha) \in U(\mathbf{K})$ , the following equalities hold

$$(3.3) \quad d^{(\ell)}(\mathcal{D}_K(x^\alpha)) = \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}),$$

$$(3.4) \quad d^{(\ell)}(a_1 \cdots a_s) = \sum_{\ell_1+\cdots+\ell_s=\ell} d^{(\ell_1)}(a_1) \cdots d^{(\ell_s)}(a_s),$$

$$(3.5) \quad \mathcal{D}_K(x^\alpha) \cdot e^m = \sum_{\ell=0}^m (-1)^\ell \binom{m}{\ell} \ell! e^{m-\ell} \cdot \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}),$$

$$(3.6) \quad (\text{ad } \mathcal{D}_K(x^\alpha))^\ell(e) = \sum_{j=0}^{\ell} \binom{\ell}{j} C_{\ell-j} D(j, k) \mathcal{D}_K(x^{\ell\alpha+2\epsilon_k+\epsilon_{-k}-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}),$$

where  $A_j = \frac{1}{j!} \prod_{i=0}^{j-1} (\alpha_k - 2\alpha_{-k} + i)$ ,  $B_{\ell-j} = \frac{1}{(\ell-j)!} \prod_{i=0}^{\ell-j-1} (i - \alpha_0)$ ,  $C_{\ell-j} = \prod_{i=0}^{\ell-j-1} (i\|\alpha\| + \alpha_0)$ ,  $D(j, k) = \prod_{i=0}^{j-1} ((2-i)\alpha_{-k} - (1-i)\alpha_k)$ ,  $A_0 = B_0 = C_0 = D(0, k) = 1$ .

PROOF. For (3.3), use induction on  $\ell$ . When  $\ell = 1$ ,

$$d(\mathcal{D}_K(x^\alpha)) = [\mathcal{D}_K(x^{2\epsilon_k+\epsilon_{-k}}), \mathcal{D}_K(x^\alpha)] = (-\alpha_0) \mathcal{D}_K(x^{\alpha+2\epsilon_k+\epsilon_{-k}-\epsilon_0}) + (\alpha_k - 2\alpha_{-k}) \mathcal{D}_K(x^{\alpha+\epsilon_k}).$$

For  $\ell \geq 1$ , we have

$$\begin{aligned} d^{(\ell+1)}(\mathcal{D}_K(x^\alpha)) &= \frac{d}{\ell+1} d^{(\ell)}(\mathcal{D}_K(x^\alpha)) \\ &= \frac{1}{\ell+1} \sum_{j=0}^{\ell} A_j B_{\ell-j} d(\mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0})) \\ &= \frac{1}{\ell+1} \sum_{j=0}^{\ell} A_j B_{\ell-j} \left( (\ell-j-\alpha_0) \mathcal{D}_K(x^{\alpha+(\ell+1)(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j+1)\epsilon_0}) \right. \\ &\quad \left. + (\alpha_k + 2\ell - j - 2(\alpha_{-k} + \ell - j)) \mathcal{D}_K(x^{\alpha+(\ell+1)(2\epsilon_k+\epsilon_{-k})-(j+1)(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) \right) \\ &= \frac{1}{\ell+1} \sum_{j=0}^{\ell} A_j (\ell-j+1) B_{\ell-j+1} \mathcal{D}_K(x^{\alpha+(\ell+1)(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j+1)\epsilon_0}) \\ &\quad + \frac{1}{\ell+1} \sum_{j=0}^{\ell} (j+1) A_{j+1} B_{\ell-j} \mathcal{D}_K(x^{\alpha+(\ell+1)(2\epsilon_k+\epsilon_{-k})-(j+1)(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) \\ &= \frac{1}{\ell+1} \sum_{j=0}^{\ell} A_j (\ell-j+1) B_{\ell-j+1} \mathcal{D}_K(x^{\alpha+(\ell+1)(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j+1)\epsilon_0}) \\ &\quad + \frac{1}{\ell+1} \sum_{j=1}^{\ell+1} j A_j B_{\ell-j+1} \mathcal{D}_K(x^{\alpha+(\ell+1)(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j+1)\epsilon_0}) \\ &= \sum_{j=0}^{\ell+1} A_j B_{\ell-j+1} \mathcal{D}_K(x^{\alpha+(\ell+1)(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell+1-j)\epsilon_0}). \end{aligned}$$

(3.4) follows from the derivation property of  $d^{(\ell)}$ .

(3.5): recalling that for any elements  $a, e$  in an associative algebra, one has

$$(3.7) \quad ca^m = \sum_{\ell=0}^m (-1)^\ell \binom{m}{\ell} a^{m-\ell} (\text{ad } a)^\ell (c).$$

Combing this with (3.4), we can get (3.5).

(3.6): Use induction on  $\ell$ . When  $\ell = 1$ , we have

$$\text{ad } \mathcal{D}_K(x^\alpha)(e) = \alpha_0 \mathcal{D}_K(x^{\alpha+2\epsilon_k+\epsilon_{-k}-\epsilon_0}) + (2\alpha_{-k} - \alpha_k) \mathcal{D}_K(x^{\alpha+\epsilon_k}).$$

When  $\ell > 1$ ,

$$\begin{aligned} (\text{ad } \mathcal{D}_K(x^\alpha))^{\ell+1}(e) &= \sum_{j=0}^{\ell} \binom{\ell}{j} C_{\ell-j} D(j, k) [\mathcal{D}_K(x^\alpha), \mathcal{D}_K(x^{\ell\alpha+2\epsilon_k+\epsilon_{-k}-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0})] \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} C_{\ell-j} D(j, k) \left( \left( (2 - \sum_{i=1}^n (\alpha_i + \alpha_{-i})) (\ell\alpha_0 - (\ell-j)) - \right. \right. \\ &\quad \left. (2-\ell \sum_{i=1}^n (\alpha_i + \alpha_{-i}) - 3+2j) \alpha_0 \right) \mathcal{D}_K(x^{(\ell+1)\alpha+2\epsilon_k+\epsilon_{-k}-j(\epsilon_k+\epsilon_{-k})-(\ell+1-j)\epsilon_0}) \\ &\quad \left. + (\alpha_{-k}(\ell\alpha_k+2-j) - \alpha_k(\ell\alpha_{-k}+1-j)) \mathcal{D}_K(x^{(\ell+1)\alpha+2\epsilon_k+\epsilon_{-k}-(j+1)(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) \right) \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} C_{\ell-j+1} D(j, k) \mathcal{D}_K(x^{(\ell+1)\alpha+2\epsilon_k+\epsilon_{-k}-j(\epsilon_k+\epsilon_{-k})-(\ell+j-1)\epsilon_0}) \\ &\quad + \sum_{j=0}^{\ell} \binom{\ell}{j} C_{\ell-j} D(j+1, k) \mathcal{D}_K(x^{(\ell+1)\alpha+2\epsilon_k+\epsilon_{-k}-(j+1)(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} C_{\ell-j+1} D(j, k) \mathcal{D}_K(x^{(\ell+1)\alpha+2\epsilon_k+\epsilon_{-k}-j(\epsilon_k+\epsilon_{-k})-(\ell-j+1)\epsilon_0}) \\ &\quad + \sum_{j=1}^{\ell+1} \binom{\ell}{j-1} C_{\ell-j+1} D(j, k) \mathcal{D}_K(x^{(\ell+1)\alpha+2\epsilon_k+\epsilon_{-k}-j(\epsilon_k+\epsilon_{-k})-(\ell+1-j)\epsilon_0}) \\ &= \sum_{j=0}^{\ell+1} \binom{\ell+1}{j} C_{\ell-j+1} D(j, k) \mathcal{D}_K(x^{(\ell+1)\alpha+2\epsilon_k+\epsilon_{-k}-j(\epsilon_k+\epsilon_{-k})-(\ell+1-j)\epsilon_0}). \end{aligned}$$

This completes the proof.  $\square$

LEMMA 3.3. For  $a \in \mathbb{F}$ ,  $\alpha \in \mathbb{Z}^{2n+1}$ , and  $\mathcal{D}_K(x^\alpha) \in K$ , the following equalities hold

$$(3.8) \quad ((\mathcal{D}_K(x^\alpha))^s \otimes 1) \cdot F_a = F_{a+s(\alpha_k-\alpha_{-k})} \cdot ((\mathcal{D}_K(x^\alpha))^s \otimes 1),$$

$$(3.9) \quad (\mathcal{D}_K(x^\alpha))^s \cdot u_a = u_{a+s(\alpha_k-\alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)} (\mathcal{D}_K(x^\alpha))^s h_{1-a}^{(\ell)} t^\ell,$$

$$(3.10) \quad (1 \otimes (\mathcal{D}_K(x^\alpha))^s) \cdot F_a = \sum_{\ell=0}^{\infty} (-1)^\ell F_{a+\ell} (h_a^{(\ell)} \otimes d^{(\ell)} (\mathcal{D}_K(x^\alpha))^s t^\ell).$$

PROOF. By (3.1) and (3.2), we have

$$\begin{aligned} ((\mathcal{D}_K(x^\alpha))^s \otimes 1) \cdot F_a &= ((\mathcal{D}_K(x^\alpha))^s \otimes 1) \left( \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes e^r t^r \right) \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} h_{a-s(\alpha_k-\alpha_{-k})}^{(m)} (\mathcal{D}_K(x^\alpha))^s \otimes e^r t^r = F_{a+s(\alpha_k-\alpha_{-k})} \cdot ((\mathcal{D}_K(x^\alpha))^s \otimes 1). \end{aligned}$$

Thus, we can obtain (3.8).

(3.9): Use induction on  $s$ . When  $s = 1$ , we have

$$\begin{aligned} \mathcal{D}_K(x^\alpha) \cdot u_a &= \mathcal{D}_K(x^\alpha) \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_{-a}^{[r]} e^r t^r \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_{-a-(\alpha_k-\alpha_{-k})}^{[r]} \sum_{\ell=0}^r (-1)^\ell \binom{r}{\ell} \ell! e^{r-\ell} \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) t^r \\ &= \sum_{r,\ell=0}^{\infty} \frac{(-1)^{r+\ell}}{(r+\ell)!} h_{-a-(\alpha_k-\alpha_{-k})}^{[r+\ell]} (-1)^\ell \binom{r+\ell}{\ell} \ell! e^r \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) t^{r+\ell} \\ &= \sum_{r,\ell=0}^{\infty} \frac{(-1)^r}{r!} h_{-a-(\alpha_k-\alpha_{-k})}^{[r]} h_{-a-(\alpha_k-\alpha_{-k})-r}^{[\ell]} e^r \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) t^{r+\ell} \\ &= \sum_{r,\ell=0}^{\infty} \frac{(-1)^r}{r!} h_{-a-(\alpha_k-\alpha_{-k})}^{[r]} e^r t^r h_{-a-(\alpha_k-\alpha_{-k})}^{[\ell]} \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) t^\ell \\ &= u_{a+(\alpha_k-\alpha_{-k})} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) h_{-a+\ell}^{[\ell]} t^\ell \\ &= u_{a+(\alpha_k-\alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha)) h_{1-a}^{(\ell)} t^\ell. \end{aligned}$$

When  $s > 1$ , we have

$$\begin{aligned} (\mathcal{D}_K(x^\alpha))^{s+1} \cdot u_a &= \mathcal{D}_K(x^\alpha) u_{a+s(\alpha_k-\alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha))^s \cdot h_{1-a}^{(\ell)} t^\ell \\ &= u_{a+(s+1)(\alpha_k-\alpha_{-k})} \sum_{\ell'=0}^{\infty} d^{(\ell')}(\mathcal{D}_K(x^\alpha)) h_{1-a-s(\alpha_k-\alpha_{-k})}^{(\ell')} t^{\ell'} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha))^s h_{1-a}^{(\ell)} t^\ell \\ &= u_{a+(s+1)(\alpha_k-\alpha_{-k})} \sum_{\ell'+\ell=0}^{\infty} d^{(\ell'+\ell)}(\mathcal{D}_K(x^\alpha))^{s+1} h_{1-a}^{(\ell'+\ell)} t^{\ell'+\ell} \\ &= u_{a+(s+1)(\alpha_k-\alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha))^{s+1} h_{1-a}^{(\ell)} t^\ell, \end{aligned}$$

where the third “=” comes from the following equation, together with (3.3) and (3.4):

$$h_{1-a-s(\alpha_k-\alpha_{-k})}^{(\ell')} \cdot d^{(\ell)}(\mathcal{D}_K(x^\alpha))^s = d^{(\ell)}(\mathcal{D}_K(x^\alpha))^s \cdot h_{1-a+\ell}^{(\ell')}.$$

(3.10): If  $s = 1$ , we have

$$\begin{aligned}
(1 \otimes \mathcal{D}_K(x^\alpha)) \cdot F_a &= (1 \otimes \mathcal{D}_K(x^\alpha)) \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes e^r t^r \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes \sum_{\ell=0}^r (-1)^\ell \binom{r}{\ell} \ell! e^{r-\ell} \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) t^r \\
&= \sum_{r,\ell=0}^{\infty} (-1)^\ell \left( \frac{1}{r!} h_a^{(r)} \otimes e^r t^r \right) (h_a^{(\ell)} \otimes \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) t^\ell) \\
&= \sum_{\ell=0}^{\infty} (-1)^\ell F_{a+\ell} h_a^{(\ell)} \otimes d^{(\ell)}(\mathcal{D}_K(x^\alpha)) t^\ell.
\end{aligned}$$

If  $s > 1$ , we have

$$\begin{aligned}
(1 \otimes (\mathcal{D}_K(x^\alpha))^{s+1}) \cdot F_a &= (1 \otimes \mathcal{D}_K(x^\alpha)) (1 \otimes (\mathcal{D}_K(x^\alpha))^s) \cdot F_a \\
&= (1 \otimes \mathcal{D}_K(x^\alpha)) \sum_{\ell=0}^{\infty} F_{a+\ell} (-1)^\ell h_a^{(\ell)} \otimes d^{(\ell)}(\mathcal{D}_K(x^\alpha))^s t^\ell \\
&= \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} (-1)^\ell F_{a+\ell+\ell'} (-1)^{\ell'} (h_{a+\ell}^{(\ell')} \otimes d^{(\ell')}(\mathcal{D}_K(x^\alpha)) t^{\ell'}) (h_a^{(\ell)} \otimes d^{(\ell)}(\mathcal{D}_K(x^\alpha))^s t^\ell) \\
&= \sum_{\ell,\ell'=0}^{\infty} (-1)^{\ell+\ell'} F_{a+\ell+\ell'} h_{a+\ell}^{(\ell')} h_a^{(\ell)} \otimes d^{(\ell')}(\mathcal{D}_K(x^\alpha)) d^{(\ell)}(\mathcal{D}_K(x^\alpha))^s t^{\ell+\ell'} \\
&= \sum_{\ell=0}^{\infty} (-1)^\ell F_{a+\ell} h_a^{(\ell)} \otimes d^{(\ell)}(\mathcal{D}_K(x^\alpha))^{s+1} t^\ell.
\end{aligned}$$

Thus, (3.10) holds by induction on  $s$ . This completes the proof.  $\square$

LEMMA 3.4. For  $s \geq 1$ , we have

$$(3.11) \quad \Delta((\mathcal{D}_K(x^\alpha))^s) = \sum_{\substack{0 \leq j \leq s \\ \ell \geq 0}} (-1)^\ell \binom{s}{j} (\mathcal{D}_K(x^\alpha))^j h^{(\ell)} \otimes (1 - et)^{j(\alpha_k - \alpha_{-k}) - \ell} (d^{(\ell)}(\mathcal{D}_K(x^\alpha))^{s-j}) t^\ell,$$

$$(3.12) \quad S((\mathcal{D}_K(x^\alpha))^s) = (-1)^s (1 - et)^{-s(\alpha_k - \alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha))^s h_1^{(\ell)} t^\ell.$$

PROOF. By Lemma 3.3, we have

$$\begin{aligned}
\Delta((\mathcal{D}_K(x^\alpha))^s) &= \mathcal{F} \sum_{j=0}^s \binom{s}{j} (\mathcal{D}_K(x^\alpha) \otimes 1)^j \mathcal{F} (1 \otimes (\mathcal{D}_K(x^\alpha))^{s-j}) \mathcal{F} \\
&= \sum_{j=0}^s \binom{s}{j} \mathcal{F} F_{j(\alpha_{-k} - \alpha_k)} (\mathcal{D}_K(x^\alpha) \otimes 1)^j (\mathcal{F} \sum_{\ell=0}^{\infty} (-1)^\ell F_\ell h^{(\ell)} \otimes d^{(\ell)}(\mathcal{D}_K(x^\alpha))^{s-j}) t^\ell \\
&= \sum_{j=0}^s \binom{s}{j} (1 \otimes (1 - et)^{j(\alpha_k - \alpha_{-k})}) (\mathcal{D}_K(x^\alpha))^j \otimes 1 \left( \sum_{\ell=0}^{\infty} (-1)^\ell (1 \otimes (1 - et)^{-\ell}) (h^{(\ell)} \otimes d^{(\ell)}(\mathcal{D}_K(x^\alpha))^{s-j}) t^\ell \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{0 \leq j \leq s \\ \ell \geq 0}} (-1)^\ell \binom{s}{j} (\mathcal{D}_K(x^\alpha))^j h^{(\ell)} \otimes (1-et)^{j(\alpha_k - \alpha_{-k}) - \ell} d^{(\ell)}(\mathcal{D}_K(x^\alpha))^{s-j} t^\ell, \\
S((\mathcal{D}_K(x^\alpha))^s) &= v(-1)^s (\mathcal{D}_K(x^\alpha))^s u = (-1)^s v u_{s(\alpha_k - \alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha))^s h_1^{(\ell)} t^\ell \\
&= (-1)^s (1-et)^{-s(\alpha_k - \alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha))^s h_1^{(\ell)} t^\ell.
\end{aligned}$$

This completes the proof.  $\square$

**THEOREM 3.5.** Fix two distinguished elements  $h = \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$ ,  $e = \mathcal{D}_K(x^{2\epsilon_k + \epsilon_{-k}})$ , such that  $[h, e] = e$  in the generalized Cartan type  $K$  Lie algebra  $\mathbf{K}$  over  $\mathbb{F}$ . There exists a structure of noncommutative and noncocommutative Hopf algebra  $(U(\mathbf{K})[[t]], m, \iota, \Delta, S, \varepsilon)$  which leaves the product of  $(U(\mathbf{K})[[t]])$  undeformed but with the deformed coproduct, the antipode and the counit defined by:

$$(3.13) \quad \Delta(\mathcal{D}_K(x^\alpha)) = \mathcal{D}_K(x^\alpha) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{\infty} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)}(\mathcal{D}_K(x^\alpha)) t^\ell,$$

$$(3.14) \quad S(\mathcal{D}_K(x^\alpha)) = -(1-et)^{\alpha_k - \alpha_{-k}} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha)) h_1^{(\ell)} t^\ell,$$

and  $\varepsilon(\mathcal{D}_K(x^\alpha)) = 0$ , for any  $\mathcal{D}_K(x^\alpha) \in \mathbf{K}$ .

**PROOF.** By Lemma 3.3, we have

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^\alpha)) &= \mathcal{F}(\mathcal{D}_K(x^\alpha) \otimes 1)F + \mathcal{F}(1 \otimes \mathcal{D}_K(x^\alpha))F \\
&= \mathcal{F}F_{\alpha_k - \alpha_{-k}}(\mathcal{D}_K(x^\alpha) \otimes 1) + \mathcal{F} \sum_{\ell=0}^{\infty} (-1)^\ell F_\ell h^{(\ell)} \otimes d^{(\ell)}(\mathcal{D}_K(x^\alpha)) t^\ell \\
&= \mathcal{D}_K(x^\alpha) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{\infty} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)}(\mathcal{D}_K(x^\alpha)) t^\ell, \\
S(\mathcal{D}_K(x^\alpha)) &= u^{-1} S_0(\mathcal{D}_K(x^\alpha))u = -v \mathcal{D}_K(x^\alpha)u = -v u_{\alpha_k - \alpha_{-k}} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha)) h_1^{(\ell)} t^\ell \\
&= -(1-et)^{-(\alpha_k - \alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha)) h_1^{(\ell)} t^\ell.
\end{aligned}$$

This completes the proof.  $\square$

Note that  $\{\mathcal{D}_K(x^\alpha) \mid \alpha \in \mathbb{Z}_+^{2n+1}\}$  is a  $\mathbb{Z}$ -basis of  $\mathbf{K}_\mathbb{Z}^+$  as a subalgebra of  $\mathbf{K}_\mathbb{Z}$  and  $\mathbf{W}_\mathbb{Z}^+$ . Consequently, we have

**COROLLARY 3.6.** Fix distinguished elements  $h = \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$ ,  $e = \mathcal{D}_K(x^{2\epsilon_k + \epsilon_{-k}})$ ,  $1 \leq k \leq n$ , the corresponding quantization of  $U(\mathbf{K}_\mathbb{Z}^+)$  over  $U(\mathbf{K}_\mathbb{Z}^+)[[t]]$  by Drinfeld's twist  $\mathcal{F}$  with the product undeformed is given by:

$$\Delta(\mathcal{D}_K(x^\alpha)) = \mathcal{D}_K(x^\alpha) \otimes (1-et)^{\alpha_k - \alpha_{-k}}$$

$$(3.15) \quad + \sum_{\ell=0}^{\infty} (-1)^{\ell} h^{(\ell)} \otimes (1-et)^{-\ell} \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) t^{\ell},$$

$$(3.16) \quad S(\mathcal{D}_K(x^{\alpha})) = -(1-et)^{\alpha_k-\alpha_k} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}) h_1^{(\ell)} t^{\ell},$$

and  $\varepsilon(\mathcal{D}_K(x^{\alpha})) = 0$ , where  $A_j = \frac{1}{j!} \prod_{i=0}^{j-1} (\alpha_k - 2\alpha_{-k} + i)$ ,  $B_{\ell-j} = \frac{1}{(\ell-j)!} \prod_{i=0}^{\ell-j-1} (i - \alpha_0)$ , with  $A_0 = B_0 = 1$ ,  $A_{-1} = B_{-1} = 0$ .

PROOF. By Theorem 3.5 and formula (3.3), we can get formula (3.15), and (3.16). By Lemma 2.7, the coefficients in the two formulas are all integers.  $\square$

**3.2. Quantization of the Contact algebra  $\mathbf{K}(2n+1; \underline{1})$ .** In this subsection, firstly, we make *modulo  $p$  reduction and base change with  $\mathcal{K}[[t]]$  replaced by  $\mathcal{K}[t]$* , for the quantization of  $U(\mathbf{K}_{\mathbb{Z}}^+)$  in characteristic 0 (Corollary 3.6) to yield the quantization of  $U(\mathbf{K}(2n+1; \underline{1}))$ , for the restricted simple modular Lie algebra  $\mathbf{K}(2n+1; \underline{1})$  in characteristic  $p$ . Secondly, we shall further make “ *$p$ -restrictedness*” reduction as well as base change with  $\mathcal{K}[t]$  replaced by  $\mathcal{K}[t]_p^{(q)}$ , for the quantization of  $U(\mathbf{K}(2n+1; \underline{1}))$ , which will lead to the required quantization of  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$ , the restricted universal enveloping algebra of  $\mathbf{K}(2n+1; \underline{1})$ .

Let  $\mathbb{Z}_p$  be the prime subfield of  $\mathcal{K}$  with  $\text{char}(\mathcal{K}) = p$ . When considering  $\mathbf{W}_{\mathbb{Z}}^+$  as a  $\mathbb{Z}_p$ -Lie algebra, namely, making modulo  $p$  reduction for the defining relations of  $\mathbf{W}_{\mathbb{Z}}^+$ , denoted by  $\mathbf{W}_{\mathbb{Z}_p}^+$ , we see that  $(J_{\underline{1}})_{\mathbb{Z}_p} = \text{Span}_{\mathbb{Z}_p} \{x^{\alpha} D_i \mid \exists j : \alpha_j \geq p\}$  is a maximal ideal of  $\mathbf{W}_{\mathbb{Z}_p}^+$ , and  $\mathbf{W}_{\mathbb{Z}_p}^+ / (J_{\underline{1}})_{\mathbb{Z}_p} \cong \mathbf{W}(2n+1; \underline{1})_{\mathbb{Z}_p} = \text{Span}_{\mathbb{Z}_p} \{x^{(\alpha)} D_i \mid 0 \leq \alpha \leq \tau, -n \leq i \leq n\}$ . For the subalgebra  $\mathbf{K}_{\mathbb{Z}}^+$ , we have  $\mathbf{K}_{\mathbb{Z}_p}^+ / (\mathbf{K}_{\mathbb{Z}_p}^+ \cap (J_{\underline{1}})_{\mathbb{Z}_p}) \cong \mathbf{K}'(2n+1; \underline{1})_{\mathbb{Z}_p}$ . We simply denote  $\mathbf{K}_{\mathbb{Z}_p}^+ \cap (J_{\underline{1}})_{\mathbb{Z}_p}$  as  $(J_{\underline{1}}^+)_{\mathbb{Z}_p}$ .

Moreover, we have  $\mathbf{K}'(2n+1; \underline{1}) = \mathcal{K} \otimes_{\mathbb{Z}_p} \mathbf{K}'(2n+1; \underline{1})_{\mathbb{Z}_p} = \mathcal{K} \mathbf{K}'(2n+1; \underline{1})_{\mathbb{Z}_p}$ , and  $\mathbf{K}_{\mathcal{K}}^+ = \mathcal{K} \mathbf{K}_{\mathbb{Z}_p}^+$ .

Observe that the ideal  $J_{\underline{1}}^+ := \mathcal{K}(J_{\underline{1}}^+)_{\mathbb{Z}_p}$  generates an ideal of  $U(\mathbf{K}_{\mathcal{K}}^+)$  over  $\mathcal{K}$ , denoted by  $J := J_{\underline{1}}^+ U(\mathbf{K}_{\mathcal{K}}^+)$ , where  $\mathbf{K}_{\mathcal{K}}^+ / J_{\underline{1}}^+ \cong \mathbf{K}'(2n+1; \underline{1})$ . Based on the formulae of Corollary 3.6,  $J$  is a Hopf ideal of  $U(\mathbf{K}_{\mathcal{K}}^+)$  satisfying  $U(\mathbf{K}_{\mathcal{K}}^+) / J \cong U(\mathbf{K}'(2n+1; \underline{1}))$ . Note that elements  $\sum a_{\alpha} \frac{1}{\alpha!} \mathcal{D}_K(x^{\alpha})$  in  $\mathbf{K}_{\mathcal{K}}^+$  for  $0 \leq \alpha \leq \tau$  will be identified with  $\sum a_{\alpha} \mathcal{D}_K(x^{(\alpha)})$  in  $\mathbf{K}'(2n+1; \underline{1})$  and those in  $J_{\underline{1}}$  with 0. Hence, by Corollary 3.6, we get the quantization of  $U(\mathbf{K}'(2n+1; \underline{1}))$  over  $U_t(\mathbf{K}'(2n+1; \underline{1})) := U(\mathbf{K}'(2n+1; \underline{1})) \otimes_{\mathcal{K}} \mathcal{K}[t]$  (not necessarily in  $U(\mathbf{K}'(2n+1; \underline{1}))[t]$ ), as seen in formulae (3.17) & (3.18) as follows.

**THEOREM 3.7.** Fix two distinguished elements  $h := \mathcal{D}_K(x^{(\epsilon_k+\epsilon_{-k})})$ ,  $e := 2\mathcal{D}_K(x^{(2\epsilon_k+\epsilon_{-k})})$  ( $1 \leq k \leq n$ ); the corresponding quantization of  $U(\mathbf{K}'(2n+1; \underline{1}))$  over  $U_t(\mathbf{K}'(2n+1; \underline{1}))$  with the product undeformed is given by

$$(3.17) \quad \Delta(\mathcal{D}_K(x^{(\alpha)})) = \mathcal{D}_K(x^{(\alpha)}) \otimes (1-et)^{\alpha_k-\alpha_k} + \sum_{\ell=0}^{p-1} \sum_{j=0}^{\ell} (-1)^{\ell} \bar{A}_{\ell} \bar{B}_{\ell-j} h^{(\ell)} \otimes (1-et)^{-\ell} \cdot \mathcal{D}_K(x^{(\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0)}) t^{\ell},$$

$$(3.18) \quad S(\mathcal{D}_K(x^{(\alpha)})) = -(1-et)^{\alpha_k - \alpha_k} \sum_{\ell=0}^{p-1} \sum_{j=0}^{\ell} \bar{A}_j \bar{B}_{\ell-j} \mathcal{D}_K(x^{(\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0)}) h_1^{(\ell)} t^\ell,$$

and  $\varepsilon(\mathcal{D}_K(x^{(\alpha)})) = 0$ , where  $\bar{A}_j = (2\ell-j)! \binom{\alpha_k+(2\ell-j)}{2\ell-j} A_j$ , with  $A_j = \frac{1}{j!} \prod_{i=0}^{j-1} (\alpha_k - 2\alpha_{-k} + i)$ ,  $A_0 = 1$ ,  $A_{-1} = 0$ . For  $0 \leq \ell-j \leq \alpha_0$ ,  $\bar{B}_{\ell-j} = (-1)^{\ell-j} \binom{\alpha_{-k}+(\ell-j)}{\ell-j}$ , otherwise,  $\bar{B}_{\ell-j} = 0$ .

PROOF. Note that the elements  $\frac{1}{\alpha!} \mathcal{D}_K(x^\alpha)$  in  $\mathbf{K}_{\mathcal{K}}^+$  will be identified with  $\mathcal{D}_K(x^{(\alpha)})$  in  $\mathbf{K}(2n+1; 1)$  and those in  $J_1$  with 0. Hence, by Corollary 3.6, we can get

$$\begin{aligned} \Delta(\mathcal{D}_K(x^{(\alpha)})) &= \frac{1}{\alpha!} \Delta(\mathcal{D}_K(x^\alpha)) \\ &= \mathcal{D}_K(x^{(\alpha)}) \otimes (1-et)^{\alpha_k - \alpha_k} + \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} \times \\ &\quad \sum_{j=0}^{\ell} A_j B_{\ell-j} \frac{(\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0)!}{\alpha!} \mathcal{D}_K(x^{(\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0)}) t^\ell, \end{aligned}$$

where

$$\begin{aligned} &A_j B_{\ell-j} \frac{(\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0)!}{\alpha!} \\ &= A_j B_{\ell-j} \frac{(\alpha_{-k}+\ell-j)! (\alpha_k+2\ell-j)! (\alpha_0-(\ell-j))!}{\alpha_{-k}! \alpha_k! \alpha_0!} \\ &= A_j \frac{(\alpha_k+2\ell-j)!}{\alpha_k!} \cdot \frac{\prod_{i=0}^{\ell-j-1} (i-\alpha_0)}{(\ell-j)!} \cdot \frac{(\alpha_{-k}+\ell-j)! (\alpha_0-(\ell-j))!}{\alpha_{-k}! \alpha_0!} \\ &= \bar{A}_j \cdot (-1)^{\ell-j} \binom{\alpha_{-k}+\ell-j}{\ell-j} = \bar{A}_j \bar{B}_{\ell-j}. \end{aligned}$$

Hence, we can get (3.17). Another formula can be obtained similarly.  $\square$

Note that for  $\alpha \leq \tau$ , if there exist  $0 < \ell \leq p-1$ ,  $0 \leq j \leq \ell$ , such that  $\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0 = \tau$ , that is,

$$\alpha_0 - (\ell-j) = p-1; \quad \alpha_k + 2\ell-j = p-1; \quad \alpha_{-k} + (\ell-j) = p-1.$$

This means that  $\alpha_0 = p-1$ ,  $j = \ell$ ,  $\alpha_k + \ell = p-1$ ,  $\alpha_{-k} = p-1$ . Thus, the coefficients of  $\mathcal{D}_K(x^{(\tau)})$  is  $\bar{A}_\ell \bar{B}_0 = \ell! \binom{\alpha_k+\ell}{\ell} A_\ell = \binom{p-1}{\ell} \prod_{i=0}^{\ell-1} (\alpha_k - 2\alpha_{-k} + i) \equiv 0 \pmod{p}$ . So whenever  $2n+4 \equiv 0 \pmod{p}$ , or  $\not\equiv 0 \pmod{p}$ , Theorem 3.7 gives the quantization of  $U(\mathbf{K}(2n+1; \underline{1}))$  over  $U_t(\mathbf{K}(2n+1; \underline{1})) = U(\mathbf{K}(2n+1; \underline{1})) \otimes_{\mathcal{K}} \mathcal{K}[t]$  and also over  $U(\mathbf{K}(2n+1; \underline{1}))[t]$ . It should be noticed that in this step including from the quantization integral form of  $U(\mathbf{K}_{\mathbb{Z}}^+)$  and making the modulo  $p$  reduction, we used the first base change with  $\mathcal{K}[[t]]$  replaced by  $\mathcal{K}[t]$ , and the objects from  $U(\mathbf{K}(2n+1; \underline{1}))[t]$  turning to  $U_t(\mathbf{K}(2n+1; \underline{1}))$ .

Assume  $0 \leq \alpha \leq \tau$  when  $2n+4 \not\equiv 0 \pmod{p}$ , and  $0 \leq \alpha < \tau$  when  $2n+4 \equiv 0 \pmod{p}$ . Denote by  $I$  the ideal of  $U(\mathbf{K}(2n+1; \underline{1}))$  over  $\mathcal{K}$  generated by  $(\mathcal{D}_K(x^{(\alpha)}))^p$  and  $(\mathcal{D}_K(x^{(\epsilon_k+\epsilon_{-k})}))^p - \mathcal{D}_K(x^{(\epsilon_k+\epsilon_{-k})})$ ,  $(\mathcal{D}_K(x^{(\epsilon_0)}))^p - \mathcal{D}_K(x^{(\epsilon_0)})$  with  $\alpha \neq \epsilon_k + \epsilon_{-k}$ ,  $\alpha \neq \epsilon_0$  for

$1 \leq k \leq n$ . Thus  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1})) = U(\mathbf{K}(2n+1; \underline{1}))/I$  is of prime-power dimension  $p^{2n+1}$  when  $2n+4 \not\equiv 0 \pmod{p}$ ; and of prime-power dimension  $p^{2n+1-1}$  otherwise. In order to get a reasonable quantization of prime-power dimension for  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$  in characteristic  $p$ , at first, it is necessary to clarify in concept what is the underlying vector space where the required  $t$ -deformed object exists. According to our modular reduction approach, it should start to be induced from the  $\mathcal{K}[t]$ -algebra  $U_t(\mathbf{K}(2n+1; \underline{1}))$  in Theorem 3.7.

Firstly, we observe the following facts (for the proof of [14] or [15]):

- LEMMA 3.8. (i)  $(1 - et)^p \equiv 1 \pmod{p, I}$ .  
(ii)  $(1 - et)^{-1} \equiv 1 + et + \cdots + e^{p-1}t^{p-1} \pmod{p, I}$ .  
(iii)  $h_a^{(\ell)} \equiv 0 \pmod{p, I}$  for  $\ell \geq p$ , and  $a \in \mathbb{Z}_p$ .

The above Lemma, together with Theorem 3.7, shows that the required  $t$ -deformation of  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$  (if it exists) in fact only happens in a  $p$ -truncated polynomial ring (with degrees of  $t$  less than  $p$ ) with coefficients in  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$ , i.e.,

$$\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})) := \mathbf{u}(\mathbf{K}(2n+1; \underline{1})) \otimes_{\mathcal{K}} \mathcal{K}[t]_p^{(q)}$$

(rather than in  $\mathbf{u}_t(\mathbf{K}(2n+1; \underline{1})) := \mathbf{u}(\mathbf{K}(2n+1; \underline{1})) \otimes_{\mathcal{K}} \mathcal{K}[t]$ ), where  $\mathcal{K}[t]_p^{(q)}$  is conveniently taken to be a  $p$ -truncated polynomial ring which is a quotient of  $\mathcal{K}[t]$  defined as

$$\mathcal{K}[t]_p^{(q)} = \mathcal{K}[t]/(t^p - qt), \quad \text{for } q \in \mathcal{K}.$$

Thereby, we obtain the underlying ring for our required  $t$ -deformation of  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$  over  $\mathcal{K}[t]_p^{(q)}$ , and

$$\dim_{\mathcal{K}} \mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})) = p \cdot \dim_{\mathcal{K}} \mathbf{u}(\mathbf{K}(2n+1; \underline{1})) = \begin{cases} p^{2n+1+1}, & 2n+4 \not\equiv 0 \pmod{p}, \\ p^{2n+1}, & 2n+4 \equiv 0 \pmod{p}. \end{cases}$$

Via modulo “ $p$ -restrictedness” reduction, it is necessary for us to work over the objects from  $U_t(\mathbf{K}(2n+1; \underline{1}))$  passage to  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$  first, and then to  $\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1}))$  (see the proof of Theorem 3.11 below), here we used the second base change with  $\mathcal{K}[t]_p^{(q)}$  instead of  $\mathcal{K}[t]$ .

As the definition of [15], we gave the following description:

DEFINITION 3.9. A Hopf algebra  $(\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})), m, \iota, \Delta, S, \varepsilon)$  over a ring  $\mathcal{K}[t]_p^{(q)}$  of characteristic  $p$  is said to be a finite-dimensional quantization of  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$  if it is obtained from the standard Hopf algebra  $U(\mathbf{K}_{\mathbb{Z}}^+)[[t]]$  by means of a Drinfel’d twisting, modular reductions and shrinking of base rings, and if there is an isomorphism as algebras  $\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1}))/t\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})) \cong \mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$ .

To describe  $\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1}))$  explicitly, we still need an auxiliary Lemma.

LEMMA 3.10. Let  $e = 2\mathcal{D}_K(x^{(2\epsilon_k + \epsilon_{-k})})$ , and  $d^{(\ell)} = \frac{1}{\ell!} \text{ad } e$ , then:

- (i)  $d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) = \sum_{j=0}^{\ell} \bar{A}_j \bar{B}_{\ell-j} \mathcal{D}_K(x^{(\alpha + \ell(2\epsilon_k + \epsilon_{-k}) - j(\epsilon_k + \epsilon_{-k}) - (\ell-j)\epsilon_0)})$ , where  $\bar{A}_j, \bar{B}_{\ell-j}$  as in Theorem 3.7.



$$\begin{aligned}
\text{(ii)} \quad & d^{(\ell)}(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) = \delta_{\ell 0} \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{\ell 1} \delta_{ik} e, \quad 1 \leq i \leq n, \\
& d^{(\ell)}(\mathcal{D}_K(x^{(\epsilon_0)})) = \delta_{\ell 0} \mathcal{D}_K(x^{(\epsilon_0)}) - \delta_{\ell 1} e. \\
\text{(iii)} \quad & d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)}))^p = \delta_{\ell 0} (\mathcal{D}_K(x^{(\alpha)}))^p - \delta_{\ell 1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) e.
\end{aligned}$$

PROOF. (i) By (3.3) and the proof of Theorem 3.7, we can get (i).

$$\text{(ii)} \quad \text{When } \ell = 1, \quad d^{(\ell)}(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) = \bar{A}_0 \bar{B}_1 \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i} + 2\epsilon_k + \epsilon_{-k} - \epsilon_0)}) + \bar{A}_1 \bar{B}_0 \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i} + \epsilon_k)}).$$

For  $\alpha = \epsilon_i + \epsilon_{-i}$ ,  $\alpha_0 = 0$ , we have:  $\bar{B}_0 = 1, \bar{B}_1 = 0$ ;  $\bar{A}_0 = 2 \binom{\delta_{ik} + 2}{2}$ ,  $\bar{A}_1 = \binom{\delta_{ik} + 1}{1} A_1 = (\delta_{ik} + 1)((\epsilon_i + \epsilon_{-i})_k - 2(\epsilon_i + \epsilon_{-i})_{-k}) = -2\delta_{ik}$ . Thus  $d(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) = -2\delta_{ik} x^{(\epsilon_i + \epsilon_{-i} + \epsilon_k)} = -\delta_{ik} e$ . Hence, we prove the first equation of (ii).

For  $\alpha = \epsilon_0$ ,  $\alpha_0 = 1$ , we can see that  $\bar{A}_0 = 2, \bar{A}_1 = 0$ ;  $\bar{B}_0 = 1, \bar{B}_1 = (-1)^1 \binom{0+1}{1} = -1$ . Thus,  $d(\mathcal{D}_K(x^{(\epsilon_0)})) = -2\mathcal{D}_K(x^{(\epsilon_0 + 2\epsilon_k + \epsilon_{-k} - \epsilon_0)}) = -e$ . This completes the second equation of (ii).

(iii) By Lemma 3.3, (3.6) and (3.7), we have

$$\begin{aligned}
& d((\mathcal{D}_K(x^{(\alpha)}))^p) = [e, (\mathcal{D}_K(x^{(\alpha)}))^p] \\
&= \sum_{\ell=1}^p (-1)^\ell \binom{p}{\ell} (\mathcal{D}_K(x^{(\alpha)}))^{p-\ell} (\text{ad } \mathcal{D}_K(x^{(\alpha)}))^\ell(e) \\
&\equiv (-1)^p (\text{ad } \mathcal{D}_K(x^{(\alpha)}))^p(e) \pmod{p} \\
&\equiv -\frac{1}{(\alpha!)^p} (\text{ad } \mathcal{D}_K(x^{(\alpha)}))^p(\mathcal{D}_K(x^{(2\epsilon_k + \epsilon_{-k})})) \pmod{p} \\
&\equiv -\frac{1}{(\alpha!)^p} \sum_{j=0}^p \binom{p}{j} C_{p-j} D(j, k) \mathcal{D}_K(x^{p\alpha + 2\epsilon_k + \epsilon_{-k} - j(\epsilon_k + \epsilon_{-k}) - (p-j)\epsilon_0}) \pmod{p} \\
&\equiv -\frac{1}{(\alpha!)^p} C_p \mathcal{D}_K(x^{p\alpha + 2\epsilon_k + \epsilon_{-k} - p\epsilon_0}) - \frac{1}{(\alpha!)^p} D(p, k) \mathcal{D}_K(x^{p\alpha + 2\epsilon_k + \epsilon_{-k} - p(\epsilon_k + \epsilon_{-k})}) \pmod{p} \\
&\equiv \begin{cases} -e, & \alpha = \epsilon_k + \epsilon_{-k}, \text{ or } \epsilon_0, \\ 0, & \text{otherwise,} \end{cases} \pmod{p, J}.
\end{aligned}$$

This completes the proof.  $\square$

Based on Theorem 3.7, Definition 3.9 and Lemma 3.10, we arrive at

**THEOREM 3.11.** Fix two distinguished elements  $h := \mathcal{D}_K(x^{(\epsilon_k + \epsilon_{-k})})$ ,  $e := 2\mathcal{D}_K(x^{(2\epsilon_k + \epsilon_{-k})})$  ( $1 \leq k \leq n$ ), there is a noncommutative and noncocommutative Hopf algebra structure  $(\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})), m, \iota, \Delta, S, \varepsilon)$  over  $\mathcal{K}[t]_p^{(q)}$  with its algebra structure undeformed, whose coalgebra structure is given by

$$(3.19) \quad \Delta(\mathcal{D}_K(x^{(\alpha)})) = \mathcal{D}_K(x^{(\alpha)}) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) t^\ell,$$

$$(3.20) \quad S(\mathcal{D}_K(x^{(\alpha)})) = -(1-et)^{\alpha_{-k} - \alpha_k} \sum_{\ell=0}^{p-1} d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) h_1^{(\ell)} t^\ell,$$

$$(3.21) \quad \varepsilon(\mathcal{D}_K(x^{(\alpha)})) = 0,$$

for  $0 \leq \alpha \leq \tau$ , which is finite dimensional with  $\dim_{\mathcal{K}\mathbf{u}_{t,q}}(\mathbf{K}(2n+1; \underline{1})) = p^{p^{2n+1}+1}$  if  $2n+4 \not\equiv 0 \pmod{p}$ , and for  $0 \leq \alpha < \tau$ , which is a finite dimensional with  $\dim_{\mathcal{K}\mathbf{u}_{t,q}}(\mathbf{K}(2n+1; \underline{1})) = p^{p^{2n+1}}$  if  $2n+4 \equiv 0 \pmod{p}$ .

PROOF. Set  $U_{t,q}(\mathbf{K}(2n+1; \underline{1})) := U(\mathbf{K}(2n+1; \underline{1})) \otimes_{\mathcal{K}} \mathcal{K}[t]_p^{(q)}$ . Note the result of Theorem 3.7, via the base change with  $\mathcal{K}[t]$  instead of  $\mathcal{K}[t]_p^{(q)}$ , is still valid over  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ . Denote by  $I_{t,q}$  the ideal of  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$  over the ring  $\mathcal{K}[t]_p^{(q)}$  generated by the same generators of the ideal  $I$  in  $U(\mathbf{K}(2n+1; \underline{1}))$  via the base change with  $\mathcal{K}$  replaced by  $\mathcal{K}[t]_p^{(q)}$ . We shall show that  $I_{t,q}$  is a Hopf ideal of  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ . It suffices to verify that  $\Delta$  and  $S$  preserve the generators in  $I_{t,q}$  of  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ .

(I) By 3.11, Lemmas 3.8, and 3.10, we have

(3.22)

$$\begin{aligned} \Delta((\mathcal{D}_K(x^{(\alpha)}))^p) &= \sum_{\substack{0 \leq j \leq p \\ \ell \geq 0}} (-1)^\ell \binom{p}{j} (\mathcal{D}_K(x^{(\alpha)}))^j h^{(\ell)} \otimes (1-et)^{j(\alpha_k - \alpha_{-k}) - \ell} d^{(\ell)} (\mathcal{D}_K(x^{(\alpha)}))^{p-j} t^\ell \\ &\equiv \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)} \cdot (\mathcal{D}_K(x^{(\alpha)}))^p t^\ell + \\ &\quad \sum_{\ell=0}^{p-1} (-1)^\ell (\mathcal{D}_K(x^{(\alpha)}))^p h^{(\ell)} \otimes (1-et)^{p(\alpha_k - \alpha_{-k}) - \ell} d^{(\ell)} \cdot 1 t^\ell \\ &\equiv 1 \otimes (\mathcal{D}_K(x^{(\alpha)}))^p + (-1)h \otimes (1-et)^{-1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) (-e)t + (\mathcal{D}_K(x^{(\alpha)}))^p \otimes 1 \\ &= 1 \otimes (\mathcal{D}_K(x^{(\alpha)}))^p + (\mathcal{D}_K(x^{(\alpha)}))^p \otimes 1 \\ &\quad + h \otimes (1-et)^{-1} \delta_{\alpha, \epsilon_i + \epsilon_{-i}} \delta_{ik} et + h \otimes (1-et)^{-1} \delta_{\alpha, \epsilon_0} et. \end{aligned}$$

If  $\alpha \neq \epsilon_i + \epsilon_{-i}, \epsilon_0$  for  $1 \leq i \leq n$ , we get

$$\begin{aligned} \Delta((\mathcal{D}_K(x^{(\alpha)}))^p) &= (\mathcal{D}_K(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (\mathcal{D}_K(x^{(\alpha)}))^p \\ &\in I_{t,q} \otimes U_{t,q}(\mathbf{K}(2n+1; \underline{1})) + U_{t,q}(\mathbf{K}(2n+1; \underline{1})) \otimes I_{t,q}. \end{aligned}$$

When  $\alpha = \epsilon_i + \epsilon_{-i}$ , by Lemma 3.10, we have

$$\Delta(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) = \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) \otimes 1 + 1 \otimes \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) + h \otimes (1-et)^{-1} \delta_{ik} et.$$

Combing this with 3.22, we have

$$\begin{aligned} \Delta((\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}))^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) &= ((\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}))^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) \otimes 1 \\ &\quad + 1 \otimes ((\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}))^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) \\ &\in I_{t,q} \otimes U_{t,q}(\mathbf{K}(2n+1; \underline{1})) + U_{t,q}(\mathbf{K}(2n+1; \underline{1})) \otimes I_{t,q}. \end{aligned}$$

When  $\alpha = \epsilon_0$ , by Lemma 3.10, we obtain

$$\Delta(\mathcal{D}_K(x^{(\epsilon_0)})) = \mathcal{D}_K(x^{(\epsilon_0)}) \otimes 1 + 1 \otimes \mathcal{D}_K(x^{(\epsilon_0)}) + h \otimes (1-et)^{-1} et.$$

Thus, we have

$$\begin{aligned} \Delta(\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) &= (\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) \otimes 1 \\ &\quad + 1 \otimes (\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) \\ &\in I_{t,q} \otimes U_{t,q}(\mathbf{K}(2n+1; \underline{1})) + U_{t,q}(\mathbf{K}(2n+1; \underline{1})) \otimes I_{t,q}. \end{aligned}$$

So the ideal  $I_{t,q}$  is a coideal of the Hopf algebra  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ .

(II) By 3.12, Lemmas 3.8 & 3.10, we have

$$\begin{aligned} S(\mathcal{D}_K(x^{(\alpha)})^p) &= (-1)^p (1-et)^{-p(\alpha_k - \alpha_{-k})} \sum_{\ell=0}^{p-1} d^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^p h_1^{(\ell)} t^\ell \\ &= -(\mathcal{D}_K(x^{(\alpha)})^p - \delta_{\alpha, \epsilon_k + \epsilon_{-k}} e h_1^{(1)} t - \delta_{\alpha, \epsilon_0} e h_1^{(1)} t) \\ &= -\mathcal{D}_K(x^{(\alpha)})^p + \delta_{\alpha, \epsilon_i + \epsilon_{-i}} \delta_{ik} e h_1^{(1)} t + \delta_{\alpha, \epsilon_0} e h_1^{(1)} t. \end{aligned}$$

When  $\alpha \neq \epsilon_i + \epsilon_{-i}$ ,  $\alpha \neq \epsilon_0$ , we have  $S(\mathcal{D}_K(x^{(\alpha)})^p) = -S(\mathcal{D}_K(x^{(\alpha)})^p) \in I_{t,q}$ . If  $\alpha = \epsilon_i + \epsilon_{-i}$ , by Lemma 3.10,  $S(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) = -(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{ik} e h_1^{(1)} t)$ . Hence,

$$\begin{aligned} S(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) &= -\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p + \delta_{ik} e h_1^{(1)} t + (\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{ik} e h_1^{(1)} t) \\ &= -(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) \in I_{t,q}. \end{aligned}$$

Using a similar argument, we have

$$\begin{aligned} S((\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)}))) &= -(\mathcal{D}_K(x^{(\epsilon_0)})^p + e h_1^{(1)} t + \mathcal{D}_K(x^{(\epsilon_0)}) - e h_1^{(1)} t) \\ &= -((\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)}))) \in I_{t,q}. \end{aligned}$$

Thus, the ideal  $I_{t,q}$  is preserved by the antipode  $S$  of the quantization  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ .

(III) It is obvious that  $\epsilon((\mathcal{D}_K(\alpha))^p) = 0$  for all  $\alpha$ .

Therefore, by (I), (II), (III), we proved that  $I_{t,q}$  is a Hopf ideal of  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ .

Thus we obtain the required  $t$ -deformation on  $\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ , for the Cartan type simple modular restricted Lie algebra of  $\mathbf{K}$  type — the Contact algebra  $\mathbf{K}(2n+1; \underline{1})$ .  $\square$

**3.3. More quantizations.** We consider the modular reduction process for the quantizations of  $U(\mathbf{K}^+)[[t]]$  arising from those products of some pairwise different and mutually commutative basic Drinfel'd twists. We will then get lots of new families of noncommutative and noncocommutative Hopf algebras of dimension  $p^{p^{2n+1}+1}$  (resp.  $p^{p^{2n+1}}$ ) with indeterminate  $t$  or of dimension  $p^{p^{2n+1}}$  (resp.  $p^{p^{2n+1}-1}$ ) with specializing  $t$  into a scalar in  $\mathcal{K}$  if  $2n+4 \not\equiv 0 \pmod{p}$  (resp. if  $2n+4 \equiv 0 \pmod{p}$ ).

Let  $A(k)_j$  and  $A(k')_j$  denote the coefficients of the corresponding quantizations of  $U(\mathbf{K}_\mathbb{Z}^+)$  over  $U(\mathbf{K}_\mathbb{Z}^+)[[t]]$  given by the Drinfel'd twists  $\mathcal{F}(k)$  and  $\mathcal{F}(k')$  as in Corollary 3.6, respectively. Note that  $A(k)_0 = A(k')_0 = 1$ ,  $A(k)_{-1} = A(k')_{-1} = 0$ .

**LEMMA 3.12.** *Fix distinguished elements  $h(k) = \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$ ,  $e(k) = \mathcal{D}_K(x^{2\epsilon_k + \epsilon_{-k}})$  ( $1 \leq k \leq n$ ) and  $h(k') = \mathcal{D}_K(x^{\epsilon_{k'} + \epsilon_{-k'}})$ ,  $e(k') = \mathcal{D}_K(x^{2\epsilon_{k'} + \epsilon_{-k'}})$  ( $1 \leq k' \leq n$ ) with  $k \neq k'$ . the corresponding quantization of  $U(\mathbf{K}_\mathbb{Z}^+)[[t]]$  by the Drinfel'd twist  $\mathcal{F} = \mathcal{F}(k)\mathcal{F}(k')$  with the product undeformed is given by*

$$\Delta(\mathcal{D}_K(x^\alpha)) = \mathcal{D}_K(x^\alpha) \otimes (1-e(k)t)^{\alpha_k - \alpha_{-k}} (1-e(k')t)^{\alpha_{k'} - \alpha_{-k'}}$$

$$\begin{aligned}
& + \sum_{n,\ell=0}^{\infty} \sum_{j'=0}^{\ell} \sum_{j=0}^n (-1)^{\ell+n} A(k)_j A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell,j'} h(k')^{(\ell)} h(k)^{(n)} \otimes (1-e(k')t)^{-\ell} (1-e(k)t)^{-n} \\
& \quad \times \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})+n(2\epsilon_k+\epsilon_{-k})-j'(\epsilon_{k'}+\epsilon_{-k'})-j(\epsilon_k+\epsilon_{-k})-(\ell+n-j-j')\epsilon_0}) t^{n+\ell}, \\
& S(\mathcal{D}_K(x^\alpha)) = -(1-e(k')t)^{\alpha_{-k'}-\alpha_{k'}} (1-e(k)t)^{\alpha_k-\alpha_{-k}} \sum_{n,\ell=0}^{\infty} \sum_{j'=0}^{\ell} \sum_{j=0}^n A(k)_j A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell,j'} \\
& \quad \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})+n(2\epsilon_k+\epsilon_{-k})-j'(\epsilon_{k'}+\epsilon_{-k'})-j(\epsilon_k+\epsilon_{-k})-(\ell+n-j-j')\epsilon_0}) \times h(k)_1^{(n)} h(k')_1^{(\ell)} t^{\ell+n}, \\
& \varepsilon(\mathcal{D}_K(x^\alpha)) = 0
\end{aligned}$$

where  $C_{n-j}^{\ell,j'} = \frac{\prod_{i=0}^{n-j-1} (i-\alpha_0+\ell-j')}{(n-j)!}$  for  $\mathcal{D}_K(x^\alpha) \in \mathbf{K}_{\mathbb{Z}}^+$ .

PROOF. First of all, let us consider the formula of comultiplication

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^\alpha)) &= \mathcal{F}(k) \mathcal{F}(k') \Delta_0(\mathcal{D}_K(x^\alpha)) \mathcal{F}(k')^{-1} \mathcal{F}(k)^{-1} \\
&= \mathcal{F}(k) \mathcal{D}_K(x^\alpha) \otimes (1-e(k')t)^{\alpha_{k'}-\alpha_{-k'}} F(k) \\
& \quad + \mathcal{F}(k) \left( \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1-e(k')t)^{-\ell} d_k^{(\ell)} \mathcal{D}_K(x^\alpha) t^\ell \right) F(k).
\end{aligned}$$

By Corollary 3.6, we can get

$$\begin{aligned}
& \mathcal{F}(k) (\mathcal{D}_K(x^\alpha) \otimes (1-e(k')t)^{\alpha_{k'}-\alpha_{-k'}}) F(k) \\
&= \mathcal{F}(k) (\mathcal{D}_K(x^\alpha) \otimes 1) (1 \otimes (1-e(k')t)^{\alpha_{k'}-\alpha_{-k'}}) F(k) \\
&= \mathcal{F}(k) (\mathcal{D}_K(x^\alpha) \otimes 1) F(k) (1 \otimes (1-e(k')t)^{\alpha_{k'}-\alpha_{-k'}}) \\
&= \mathcal{F}(k) F(k)_{\alpha_k-\alpha_{-k}} (\mathcal{D}_K(x^\alpha) \otimes 1) (1 \otimes (1-e(k')t)^{\alpha_{k'}-\alpha_{-k'}}) \\
&= \mathcal{D}_K(x^\alpha) \otimes (1-e(k)t)^{\alpha_k-\alpha_{-k}} (1-e(k')t)^{\alpha_{k'}-\alpha_{-k'}}, \\
& \mathcal{F}(k) \left( \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1-e(k')t)^{-\ell} d_k^{(\ell)} (\mathcal{D}_K(x^\alpha)) t^\ell \right) F(k) \\
&= \mathcal{F}(k) \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1-e(k')t)^{-\ell} \times \\
& \quad \times \sum_{j'=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})-j'(\epsilon_{k'}+\epsilon_{-k'})-(\ell-j')\epsilon_0}) t^\ell F(k) \\
&= \left( \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1-e(k')t)^{-\ell} \right) \times \\
& \quad \times \left( \sum_{j'=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} \mathcal{F}(k) (1 \otimes \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})-j'(\epsilon_{k'}+\epsilon_{-k'})-(\ell-j')\epsilon_0})) F(k) t^\ell \right) \\
&= \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1-e(k')t)^{-\ell} \sum_{j'=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} \mathcal{F}(k) \times \\
& \quad \times \sum_{n=0}^{\infty} (-1)^n F(k)_n \left( h(k)^{(n)} \otimes d_k^{(n)} (\mathcal{D}_K(x^{\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})-j'(\epsilon_{k'}+\epsilon_{-k'})-(\ell-j')\epsilon_0})) \right) t^{n+\ell}.
\end{aligned}$$

Set  $\alpha(\ell, k', j') := \alpha + \ell(2\epsilon_{k'} + \epsilon_{-k'}) - j'(\epsilon_{k'} + \epsilon_{-k'}) - (\ell - j')\epsilon_0$ . It is easy to see

$$\begin{aligned}
d_k^{(n)} \mathcal{D}_K(x^{\alpha + \ell(2\epsilon_{k'} + \epsilon_{-k'}) - j'(\epsilon_{k'} + \epsilon_{-k'}) - (\ell - j')\epsilon_0}) &= d_k^{(n)} \mathcal{D}_K(x^{\alpha(\ell, k', j')}) \\
&= \sum_{j=0}^n \frac{\prod_{i=0}^{n-j-1} (i - \alpha(\ell, k', j')_0)}{(n-j)!} \frac{\prod_{i=0}^{j-1} (\alpha(\ell, k', j')_k - 2\alpha(\ell, k', j')_{-k} + i)}{j!} \times \\
&\quad \times \mathcal{D}_K(x^{\alpha(\ell, k', j') + n(2\epsilon_k + \epsilon_{-k}) - j(\epsilon_k + \epsilon_{-k}) - (n-j)\epsilon_0}) \\
&= \sum_{j=0}^n \frac{\prod_{i=0}^{n-j-1} (i - \alpha_0 + \ell - j')}{(n-j)!} \frac{\prod_{i=0}^{j-1} (\alpha_k - 2\alpha_{-k} + i)}{j!} \times \\
&\quad \times \mathcal{D}_K(x^{\alpha + \ell(2\epsilon_{k'} + \epsilon_{-k'}) - j'(\epsilon_{k'} + \epsilon_{-k'}) - (\ell - j')\epsilon_0 + n(2\epsilon_k + \epsilon_{-k}) - j(\epsilon_k + \epsilon_{-k}) - (n-j)\epsilon_0}) \\
&= \sum_{j=0}^n C_{n-j}^{\ell, j'} A(k)_j \mathcal{D}_K(x^{\alpha + \ell(2\epsilon_{k'} + \epsilon_{-k'}) + n(2\epsilon_k + \epsilon_{-k}) - j'(\epsilon_{k'} + \epsilon_{-k'}) - j(\epsilon_k + \epsilon_{-k}) - (\ell + n - j' - j)\epsilon_0}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{F}(k) &\left( \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1 - e(k')t)^{-\ell} d_{k'}^{(\ell)} \mathcal{D}_K(x^\alpha) t^\ell \right) F(k) \\
&= \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1 - e(k')t)^{-\ell} \sum_{j'=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} \sum_{n=0}^{\infty} (-1)^n (1 \otimes (1 - e(k)t)^{-n}) \\
&\quad \left( h(k)^{(n)} \otimes \sum_{j=0}^n C_{n-j}^{\ell, j'} A(k)_j \mathcal{D}_K(x^{\alpha + \ell(2\epsilon_{k'} + \epsilon_{-k'}) - j'(\epsilon_{k'} + \epsilon_{-k'}) - (\ell - j')\epsilon_0 + n(2\epsilon_k + \epsilon_{-k}) - j(\epsilon_k + \epsilon_{-k}) - (n-j)\epsilon_0}) \right) t^{n+\ell} \\
&= \sum_{n, \ell=0}^{\infty} \sum_{j'=0}^{\ell} \sum_{j=0}^n (-1)^{\ell+n} h(k')^{(\ell)} h(k)^{(n)} \otimes (1 - e(k')t)^{-\ell} (1 - e(k)t)^{-n} A(k)_j A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell, j'} \\
&\quad \cdot \mathcal{D}_K(x^{\alpha + \ell(2\epsilon_{k'} + \epsilon_{-k'}) + n(2\epsilon_k + \epsilon_{-k}) - j'(\epsilon_{k'} + \epsilon_{-k'}) - j(\epsilon_k + \epsilon_{-k}) - (\ell + n - j' - j)\epsilon_0}) t^{n+\ell}.
\end{aligned}$$

So we get the required formula for the comultiplication.

Next, we consider the second formula

$$\begin{aligned}
S(\mathcal{D}_K(x^\alpha)) &= -v(k)v(k') \mathcal{D}_K(x^\alpha) u(k') u(k) \\
&= -v(k)(1 - e(k')t)^{\alpha_{-k'} - \alpha_{k'}} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} \mathcal{D}_K(x^{\alpha + \ell(2\epsilon_{k'} + \epsilon_{-k'}) - j'(\epsilon_{k'} + \epsilon_{-k'}) - (\ell - j')\epsilon_0}) h(k')_1^{(\ell)} u(k) t^\ell \\
&= -(1 - e(k')t)^{\alpha_{-k'} - \alpha_{k'}} \sum_{\ell=0}^{\infty} \sum_{j'=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} v(k) u(k)_{\alpha_k - \alpha_{-k}} \times \\
&\quad \times \sum_{n=0}^{\infty} d_k^{(n)} \mathcal{D}_K(x^{\alpha + \ell(2\epsilon_{k'} + \epsilon_{-k'}) - j'(\epsilon_{k'} + \epsilon_{-k'}) - (\ell - j')\epsilon_0}) h(k)_1^{(n)} t^n h(k')_1^{(\ell)} t^\ell \\
&= -(1 - e(k')t)^{\alpha_{-k'} - \alpha_{k'}} (1 - e(k)t)^{\alpha_{-k} - \alpha_k} \sum_{\ell=0}^{\infty} \sum_{j'=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} \sum_{n=0}^{\infty} \sum_{j=0}^n C_{n-j}^{\ell, j'} A(k)_j \times \\
&\quad \times \mathcal{D}_K(x^{\alpha + \ell(2\epsilon_{k'} + \epsilon_{-k'}) - j'(\epsilon_{k'} + \epsilon_{-k'}) - (\ell - j')\epsilon_0 + n(2\epsilon_k + \epsilon_{-k}) - j(\epsilon_k + \epsilon_{-k}) - (n-j)\epsilon_0}) h(k)_1^{(n)} h(k')_1^{(\ell)} t^{n+\ell}
\end{aligned}$$

$$\begin{aligned}
&= -(1-e(k')t)^{\alpha_{-k'}-\alpha_{k'}} (1-e(k)t)^{\alpha_{-k}-\alpha_k} \sum_{n,\ell=0}^{\infty} \sum_{j'=0}^{\ell} \sum_{j=0}^n A(k)_j A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell,j'} \times \\
&\quad \times \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})+n(2\epsilon_k+\epsilon_{-k})-j'(\epsilon_{k'}+\epsilon_{-k'})-j(\epsilon_k+\epsilon_{-k})-(\ell+n-j-j')\epsilon_0}) h(k)_1^{\langle n \rangle} h(k')_1^{\langle \ell \rangle} t^{n+\ell}.
\end{aligned}$$

This completes the proof.  $\square$

LEMMA 3.13. Fix distinguished elements  $h(k) = \mathcal{D}_K(x^{\epsilon_k+\epsilon_{-k}})$ ,  $e(k) = 2\mathcal{D}_K(x^{(2\epsilon_k+\epsilon_{-k})})$ ,  $h(k') = \mathcal{D}_K(x^{\epsilon_{k'}+\epsilon_{-k'}})$ ,  $e(k') = 2\mathcal{D}_K(x^{(2\epsilon_{k'}+\epsilon_{-k'})})$ , with  $1 \leq k \neq k' \leq n$ , then the associated quantization of  $U(\mathbf{K}(2n+1; \underline{1}))$  on  $U_t(\mathbf{K}(2n+1; \underline{1}))$  (also on  $U(\mathbf{K}(2n+1; \underline{1}))[[t]]$ ) with the product undeformed is given by

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^{(\alpha)})) &= \mathcal{D}_K(x^{(\alpha)}) \otimes (1-e(k)t)^{\alpha_{-k}-\alpha_k} (1-e(k')t)^{\alpha_{k'}-\alpha_{-k'}} \\
&+ \sum_{n,\ell=0}^{p-1} (-1)^{\ell+n} \sum_{j'=0}^{\ell} \sum_{j=0}^n \overline{A(k', \ell)}_{j'} \overline{A(k, n)}_j \overline{B(k, k')_{n,j}^{\ell,j'}} h(k')^{\langle \ell \rangle} h(k)^{\langle n \rangle} \otimes (1-e(k')t)^{-\ell} (1-e(k)t)^{-n} \\
&\quad \times \mathcal{D}_K(x^{(\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})+n(2\epsilon_k+\epsilon_{-k})-j'(\epsilon_{k'}+\epsilon_{-k'})-j(\epsilon_k+\epsilon_{-k})-(\ell+n-j-j')\epsilon_0)}) t^{n+\ell}, \\
S(\mathcal{D}_K(x^{(\alpha)})) &= -(1-e(k')t)^{\alpha_{-k'}-\alpha_{k'}} (1-e(k)t)^{\alpha_{-k}-\alpha_k} \sum_{n,\ell=0}^{p-1} \sum_{j'=0}^{\ell} \sum_{j=0}^n \overline{A(k', \ell)}_{j'} \overline{A(k, n)}_j \overline{B(k, k')_{n,j}^{\ell,j'}} \\
&\quad \times \mathcal{D}_K(x^{(\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})+n(2\epsilon_k+\epsilon_{-k})-j'(\epsilon_{k'}+\epsilon_{-k'})-j(\epsilon_k+\epsilon_{-k})-(\ell+n-j-j')\epsilon_0)}) h(k)_1^{\langle n \rangle} h(k')_1^{\langle \ell \rangle} t^{\ell+n}, \\
\varepsilon(\mathcal{D}_K(x^{(\alpha)})) &= 0,
\end{aligned}$$

where  $0 \leq \alpha \leq \tau$  for  $2n+4 \not\equiv 0 \pmod{p}$ , and  $0 \leq \alpha < \tau$  for  $2n+4 \equiv 0 \pmod{p}$ .  $\overline{A(k', \ell)}_{j'} = (2\ell-j')! \binom{\alpha_{k'}+2\ell-j'}{2\ell-j'}$ ,  $\overline{A(k, n)}_j = (2n-j)! \binom{\alpha_k+2n-j}{2n-j}$ ,  $\overline{B(k, k')_{n,j}^{\ell,j'}} = (-1)^{\ell+n-j-j'} \binom{\alpha_k+\ell-j}{\ell-j} \binom{\alpha_{-k'}+\ell-j'}{\ell-j'}$  for  $n+\ell-j-j' \leq \alpha_0$ , otherwise  $\overline{B(k, k')_{n,j}^{\ell,j'}} = 0$ .

PROOF. Note that the elements  $\frac{1}{\alpha!} \mathcal{D}_K(x^{(\alpha)})$  in  $\mathbf{K}_K^+$  will be identified with  $\mathcal{D}_K(x^{(\alpha)})$  in  $\mathbf{K}(2n+1; 1)$  and those in  $J_{\underline{1}}$  with 0. Hence, by Lemma 3.12, we can get

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^{(\alpha)})) &= \frac{1}{\alpha!} \Delta(\mathcal{D}_K(x^{(\alpha)})) = \mathcal{D}_K(x^{(\alpha)}) \otimes (1-e(k)t)^{\alpha_{-k}-\alpha_k} (1-e(k')t)^{\alpha_{k'}-\alpha_{-k'}} \\
&+ \sum_{n,\ell=0}^{p-1} h(k')^{\langle \ell \rangle} h(k)^{\langle n \rangle} \otimes (1-e(k')t)^{-\ell} (1-e(k)t)^{-n} \sum_{j'=0}^{\ell} \sum_{j=0}^n A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell,j'} A(k)_j \\
&\quad \cdot \frac{(\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})+n(2\epsilon_k+\epsilon_{-k})-j'(\epsilon_{k'}+\epsilon_{-k'})-j(\epsilon_k+\epsilon_{-k})-(n+\ell-j-j')\epsilon_0)!}{\alpha!} \\
&\quad \cdot \mathcal{D}_K(x^{(\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})+n(2\epsilon_k+\epsilon_{-k})-j'(\epsilon_{k'}+\epsilon_{-k'})-j(\epsilon_k+\epsilon_{-k})-(n+\ell-j-j')\epsilon_0)}) t^{n+\ell}.
\end{aligned}$$

We can see that

$$\begin{aligned}
&A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell,j'} A(k)_j \frac{(\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})+n(2\epsilon_k+\epsilon_{-k})-j'(\epsilon_{k'}+\epsilon_{-k'})-j(\epsilon_k+\epsilon_{-k})-(n+\ell-j-j')\epsilon_0)!}{\alpha!} \\
&= A(k')_{j'} \frac{\prod_{i=0}^{\ell-j'-1} (i-\alpha_0)}{(\ell-j')!} \frac{\prod_{i=0}^{n-j-1} (i-\alpha_0+\ell-j')}{(n-j)!} A(k)_j \cdot \\
&\quad \cdot \frac{(\alpha_{k'}+2\ell-j')! (\alpha_{-k'}+\ell-j')! (\alpha_k+2n-j)! (\alpha_{-k}+n-j)! (\alpha_0-(\ell+n-j-j'))!}{\alpha_{k'}! \alpha_{-k'}! \alpha_k! \alpha_{-k}! \alpha_0!}
\end{aligned}$$

$$\begin{aligned}
&= A(k')_{j'} \frac{(\alpha_{k'}+2\ell-j')!}{\alpha_{k'}!} A(k)_j \frac{(\alpha_k+2n-j)!}{\alpha_k!} \frac{\prod_{i=0}^{\ell-j'-1} (i-\alpha_0)}{(\ell-j')!} \frac{\prod_{i=0}^{n-j-1} (i-\alpha_0+\ell-j')}{(n-j)!} \\
&\quad \cdot \frac{(\alpha_{-k'}+\ell-j')!}{\alpha_{-k'}!} \frac{(\alpha_0-(\ell+n-j-j'))!}{\alpha_0!} \frac{(\alpha_{-k}+n-j)!}{\alpha_{-k}!} \\
&= A(k')_{j'} \frac{(\alpha_{k'}+2\ell-j')!}{2\ell-j'} A(k)_j \frac{(\alpha_k+2n-j)!}{2n-j} \binom{\alpha_{-k'}+\ell-j'}{\ell-j'} \binom{\alpha_{-k}+n-j}{n-j} \\
&\quad \cdot \frac{(-1)^{\ell-j'} \alpha_0 \cdots (\alpha_0-(\ell-j')+1) (-1)^{n-j} (\alpha_0-(\ell-j')) \cdots (\alpha_0-(\ell+n-j-j'))!}{\alpha_0!} \\
&= \overline{A(k', \ell)}_{j'} \overline{A(k, n)}_j \binom{\alpha_{-k'}+\ell-j'}{\ell-j'} \binom{\alpha_{-k}+n-j}{n-j} B,
\end{aligned}$$

where

$$B = \frac{(-1)^{\ell+n-j-j'} \alpha_0 \cdots (\alpha_0-(\ell-j')+1) (\alpha_0-(\ell-j')) \cdots (\alpha_0-(\ell+n-j-j'))!}{\alpha_0!} = (-1)^{\ell+n-j-j'},$$

when  $\ell-j' \leq \alpha_0$ , and  $n-j \leq \alpha_0-(\ell-j')$ . In fact, it is easy to see  $0 \leq n-j \leq \alpha_0-(\ell-j')$  implies  $\ell-j' \leq \alpha_0$ . So we have  $B = (-1)^{\ell+n-j-j'}$ , if  $n-j \leq \alpha_0-(\ell-j')$ , i.e.  $n+\ell-j-j' \leq \alpha_0$ . Otherwise,  $B = 0$ .

$$\begin{aligned}
&A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell, j'} A(k)_j \times \\
&\quad \times \frac{(\alpha+\ell(2\epsilon_{k'}+\epsilon_{-k'})+n(2\epsilon_k+\epsilon_{-k})-j'(\epsilon_{k'}+\epsilon_{-k'})-j(\epsilon_k+\epsilon_{-k})-(n+\ell-j-j')\epsilon_0)!}{\alpha!} \\
&= \overline{A(k', \ell)}_{j'} \overline{A(k, n)}_j \binom{\alpha_{-k'}+\ell-j'}{\ell-j'} \binom{\alpha_{-k}+n-j}{n-j} B \\
&= \overline{A(k', \ell)}_{j'} \overline{A(k, n)}_j \overline{B(k, k')}_{n, j}^{\ell, j'}.
\end{aligned}$$

Hence, we get the first formula.

Applying a similar argument, we can get the other formulas.

This completes the proof.  $\square$

LEMMA 3.14. *For  $s \geq 1$ , one has*

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^{(\alpha)})^s) &= \sum_{\substack{0 \leq j' \leq s \\ n, \ell \geq 0}} \binom{s}{j'} (-1)^{n+\ell} \mathcal{D}_K(x^{(\alpha)})^{j'} h(k')^{(\ell)} h(k)^{(n)} \otimes \\
&\quad (1-e(k)t)^{j'(\alpha_k-\alpha_{-k})-n} (1-e(k')t)^{j(\alpha_{k'}-\alpha_{-k'})-\ell} d_k^{(n)} (d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^{s-j'}) t^{n+\ell}, \\
S(\mathcal{D}_K(x^{(\alpha)})^s) &= (-1)^s (1-e(k')t)^{-s(\alpha_{k'}-\alpha_{-k'})} (1-e(k)t)^{-s(\alpha_k-\alpha_{-k})} \times \\
&\quad \times \sum_{n, \ell=0}^{\infty} d_k^{(n)} (d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^s) h(k')_1^{(\ell)} h(k)_1^{(n)} t^{n+\ell}.
\end{aligned}$$

PROOF. Let us calculate the following

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^{(\alpha)})^s) &= \mathcal{F}(k)(\mathcal{D}_K(x^{(\alpha)}) \otimes 1 + 1 \otimes \mathcal{D}_K(x^{(\alpha)})^s \mathcal{F}^{-1}(k) \\
&= \mathcal{F}(k) \sum_{\substack{0 \leq j' \leq s \\ \ell \geq 0}} \binom{s}{j'} (-1)^{\ell} \mathcal{D}_K(x^{(\alpha)})^{j'} h(k')^{(\ell)} \otimes (1-e(k')t)^{j'(\alpha_{k'}-\alpha_{-k'})-\ell} d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^{s-j'} t^{\ell} F(k)
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{F}(k) \sum_{\substack{0 \leq j' \leq s \\ \ell \geq 0}} \binom{s}{j'} (-1)^\ell (\mathcal{D}_K(x^{(\alpha)})^{j'} \otimes 1) (h(k')^{\langle \ell \rangle} \otimes (1-e(k')t)^{j'(\alpha_{k'}-\alpha_{-k'})-\ell}) \\
&\quad \cdot (1 \otimes d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^{s-j'}) F(k) t^\ell \\
&= \mathcal{F}(k) \sum_{\substack{0 \leq j' \leq s \\ \ell \geq 0}} \binom{s}{j'} (-1)^\ell (\mathcal{D}_K(x^{(\alpha)})^{j'} \otimes 1) (h(k')^{\langle \ell \rangle} \otimes (1-e(k')t)^{j'(\alpha_{k'}-\alpha_{-k'})-\ell}) \\
&\quad \sum_{n=0}^{\infty} (-1)^n F(k)_n h(k)^{\langle n \rangle} \otimes d_k^{(n)} (d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^{s-j'}) t^n t^\ell \\
&= \mathcal{F}(k) \sum_{\substack{0 \leq j' \leq s \\ \ell \geq 0}} \binom{s}{j'} (-1)^\ell (\mathcal{D}_K(x^{(\alpha)})^{j'} \otimes 1) F(k)_n \\
&\quad \sum_{n=0}^{\infty} (-1)^n h(k')^{\langle \ell \rangle} h(k)^{\langle n \rangle} \otimes (1-e(k')t)^{j'(\alpha_{k'}-\alpha_{-k'})-\ell} d_k^{(n)} (d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^{s-j'}) t^{n+\ell} \\
&= \sum_{\substack{0 \leq j' \leq s \\ \ell \geq 0}} \binom{s}{j'} (-1)^\ell \mathcal{F}(k) F(k)_{n+j'(\alpha_{-k}-\alpha_k)} (\mathcal{D}_K(x^{(\alpha)})^{j'} \otimes 1) \\
&\quad \sum_{n=0}^{\infty} (-1)^n h(k')^{\langle \ell \rangle} h(k)^{\langle n \rangle} \otimes (1-e(k')t)^{j'(\alpha_{k'}-\alpha_{-k'})-\ell} d_k^{(n)} (d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^{s-j'}) t^{n+\ell} \\
&= \sum_{\substack{0 \leq j' \leq s \\ n, \ell \geq 0}} \binom{s}{j'} (-1)^{n+\ell} \mathcal{D}_K(x^{(\alpha)})^{j'} h(k')^{\langle \ell \rangle} h(k)^{\langle n \rangle} \otimes \\
&\quad (1-e(k)t)^{j'(\alpha_k-\alpha_{-k})-n} (1-e(k')t)^{j'(\alpha_{k'}-\alpha_{-k'})-\ell} d_k^{(n)} (d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^{s-j'}) t^{n+\ell}, \\
&S((\mathcal{D}_K(x^{(\alpha)}))^s) = v(k)v(k')S_0(\mathcal{D}_K(x^{(\alpha)})^s)u(k')u(k) \\
&= (-1)^s v(k)(1-e(k')t)^{-s(\alpha_{k'}-\alpha_{-k'})} \sum_{\ell=0}^{\infty} d_{k'}^{(\ell)} (\mathcal{D}_K(x^{(\alpha)})^s) h(k')_1^{\langle \ell \rangle} t^\ell u(k) \\
&= (-1)^s v(k)(1-e(k')t)^{-s(\alpha_{k'}-\alpha_{-k'})} u(k)_{s(\alpha_k-\alpha_{-k})} \sum_{\ell, n=0}^{\infty} d_k^{(n)} (d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^s) h(k)_1^{\langle n \rangle} h(k')_1^{\langle \ell \rangle} t^{n+\ell} \\
&= (-1)^s (1-e(k')t)^{-s(\alpha_{k'}-\alpha_{-k'})} (1-e(k)t)^{-s(\alpha_k-\alpha_{-k})} \sum_{n, \ell} d_k^{(n)} (d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^s) h(k')_1^{\langle \ell \rangle} h(k)_1^{\langle n \rangle} t^{n+\ell}.
\end{aligned}$$

This completes the proof.  $\square$

To describe  $\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1}))$  explicitly, we still need an auxiliary Lemma.

**LEMMA 3.15.** *Set  $e(k) = 2\mathcal{D}_K(x^{(2\epsilon_k+\epsilon_{-k})})$ ,  $e(k') = 2\mathcal{D}_K(x^{(2\epsilon_{k'}+\epsilon_{-k'})})$ ,  $d_k^{(n)} = \frac{1}{n!}(\text{ad } e(k))^n$ ,  $d_{k'}^{(\ell)} = \frac{1}{\ell!}(\text{ad } e(k'))^\ell$ , Then*

$$\begin{aligned}
\text{(i)} \quad & d_k^{(n)} d_{k'}^{(\ell)} (\mathcal{D}_K(x^{(\alpha)})) = \sum_{j'=0}^{\ell} \sum_{j=0}^n \overline{A(k', \ell)}_j \overline{A(k, n)}_j \overline{B(k, k')}_{n,j}^{\ell, j'} \times \\
& \quad \times \mathcal{D}_K(x^{(\alpha+(\ell(2\epsilon_{k'}+\epsilon_{-k'})+n(2\epsilon_k+\epsilon_{-k})-j'(\epsilon_{k'}+\epsilon_{-k'})-j(\epsilon_k+\epsilon_{-k})-(\ell+n-j-j')\epsilon_0))}). \\
\text{(ii)} \quad & d_k^{(n)} d_{k'}^{(\ell)} (\mathcal{D}_K(x^{(\epsilon_j+\epsilon_{-i})})) = \delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\epsilon_j+\epsilon_{-i})}) - \delta_{\ell 1} \delta_{n 0} \delta_{ik'} e(k') - \delta_{\ell 0} \delta_{n 1} \delta_{ik} e(k),
\end{aligned}$$



$$\begin{aligned}
d_k^{(n)} d_{k'}^{(\ell)} (\mathcal{D}_K(x^{(\epsilon_0)})) &= \delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\epsilon_0)}) - \delta_{\ell 0} \delta_{n 1} e(k) - \delta_{\ell 1} \delta_{n 0} e(k'). \\
\text{(iii)} \quad d_k^{(n)} d_{k'}^{(\ell)} ((\mathcal{D}_K(x^{(\alpha)}))^p) &= \delta_{\ell 0} \delta_{n 0} \mathcal{D}_K((x^{(\alpha)})^p) - \delta_{\ell 0} \delta_{n 1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) e(k) \\
&\quad - \delta_{\ell 1} \delta_{n 0} (\delta_{\alpha, \epsilon_{k'} + \epsilon_{-k'}} + \delta_{\alpha, \epsilon_0}) e(k').
\end{aligned}$$

PROOF. (i) follows from the proof of Lemma 3.10.

(ii) By Lemma 3.10 (ii), we have

$$\begin{aligned}
d_k^{(n)} d_{k'}^{(\ell)} (\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) &= d_k^{(n)} (\delta_{\ell 0} \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{\ell 1} \delta_{ik'} e(k')) \\
&= \delta_{\ell 0} d_k^{(n)} (\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) - \delta_{\ell 1} \delta_{ik'} d_k^{(n)} \cdot (e(k')) \\
&= \delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{\ell 0} \delta_{n 1} \delta_{ik} e(k) - \delta_{\ell 1} \delta_{n 0} \delta_{ik'} e(k'), \\
d_k^{(n)} d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\epsilon_0)}) &= d_k^{(n)} (\delta_{\ell 0} \mathcal{D}_K(x^{(\epsilon_0)}) - \delta_{\ell 1} e(k')) \\
&= \delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\epsilon_0)}) - \delta_{\ell 0} \delta_{n 1} e(k) - \delta_{n 0} \delta_{\ell 1} e(k').
\end{aligned}$$

(iii) By (iii) of Lemma 3.10, we get

$$\begin{aligned}
d_k^{(n)} d_{k'}^{(\ell)} (\mathcal{D}_K(x^{(\alpha)}))^p &= d_k^{(n)} (\delta_{\ell 0} (\mathcal{D}_K(x^{(\alpha)}))^p - \delta_{\ell 1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) e(k)) \\
&= \delta_{\ell 0} \delta_{n 0} (\mathcal{D}_K(x^{(\alpha)}))^p - \delta_{\ell 0} \delta_{n 1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) e(k) - \delta_{n 0} \delta_{\ell 1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) e(k').
\end{aligned}$$

We complete the proof.  $\square$

Using Lemmas 3.12, 3.13, 3.14 and 3.15, we get a new Hopf algebra structure over the same restricted universal enveloping algebra  $\mathbf{u}(\mathbf{K}(2n+1; \underline{1}))$  over  $\mathcal{K}$  by the product of two different and commutative basic Drinfel'd twists.

**THEOREM 3.16.** *Fix distinguished elements  $h(k) := \mathcal{D}_K(x^{(\epsilon_k + \epsilon_{-k})})$ ,  $e(k) = 2\mathcal{D}_K(x^{(2\epsilon_k + \epsilon_{-k})})$ ,  $h(k') = \mathcal{D}_K(x^{(\epsilon_{k'} + \epsilon_{-k'})})$ ,  $e(k') = 2\mathcal{D}_K(x^{(2\epsilon_{k'} + \epsilon_{-k'})})$ , with  $1 \leq k \neq k' \leq n$ , there is a noncommutative and noncocommutative Hopf algebra  $(\mathbf{u}_{t,q}(\mathbf{K})(2n+1; \underline{1}), m, \iota, \Delta, S, \varepsilon)$  over  $\mathcal{K}[t]_p^{(q)}$  with the product undeformed, whose coalgebra structure is given by*

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^{(\alpha)})) &= \mathcal{D}_K(x^{(\alpha)}) \otimes (1 - e(k)t)^{\alpha_k - \alpha_{-k}} (1 - e(k')t)^{\alpha_{k'} - \alpha_{-k'}} \\
&\quad + \sum_{n, \ell=0}^{p-1} (-1)^{\ell+n} h(k')^{(\ell)} h(k)^{(n)} \otimes (1 - e(k')t)^{-\ell} (1 - e(k)t)^{-n} d_k^{(n)} d_{k'}^{(\ell)} (\mathcal{D}_K(x^{(\alpha)})) t^{n+\ell}, \\
S(x^{(\alpha)}) &= -(1 - e(k')t)^{\alpha_{-k'} - \alpha_{k'}} (1 - e(k)t)^{\alpha_{-k} - \alpha_k} \sum_{n, \ell=0}^{p-1} d_k^{(n)} d_{k'}^{(\ell)} (\mathcal{D}_K(x^{(\alpha)})) h(k)_1^{(n)} h(k')_1^{(\ell)} t^{n+\ell}, \\
\varepsilon(\mathcal{D}_K(x^{(\alpha)})) &= 0,
\end{aligned}$$

for  $0 \leq \alpha \leq \tau$ , and  $\dim_{\mathcal{K}} \mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})) = p^{p^{2n+1}+1}$ , if  $2n+4 \not\equiv 0 \pmod{p}$ ; for  $0 \leq \alpha < \tau$ ,  $\dim_{\mathcal{K}} \mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})) = p^{p^{2n+1}}$ , if  $2n+4 \equiv 0 \pmod{p}$ .

PROOF. Let  $I_{t,q}$  denote the ideal of  $(U_{t,q}(\mathbf{K}(2n+1; \underline{1})))$  over the ring  $\mathcal{K}[t]_p^{(q)}$  generated by the same generators as in  $I$  ( $q \in \mathcal{K}$ ). Observe that the result in Lemma 3.8, via the base change with  $\mathcal{K}[t]$  replaced by  $\mathcal{K}[t]_p^{(q)}$ , the deformation is still valid for  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ .

In what follows, we shall show that  $I_{t,q}$  is a Hopf ideal of  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ . To this end, it suffices to verify that  $\Delta$  and  $S$  preserve the generators of  $I_{t,q}$ .

(I) By Lemmas 3.8, 3.14 & 3.15, we have

(3.23)

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^{(\alpha)})^p) &= \sum_{\substack{0 \leq j' \leq p \\ n, \ell \geq 0}} \binom{p}{j'} (-1)^{n+\ell} \mathcal{D}_K(x^{(\alpha)})^{j'} h(k')^{(\ell)} h(k)^{(n)} \otimes (1-e(k)t)^{j'(\alpha_k - \alpha_{-k}) - n} \times \\
&\quad \times (1-e(k')t)^{j'(\alpha_{k'} - \alpha_{-k'}) - \ell} d_k^{(n)} d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^{p-j'} t^{n+\ell} \\
&\equiv \sum_{n, \ell=0}^{\infty} (-1)^{n+\ell} h(k')^{(\ell)} h(k)^{(n)} \otimes (1-e(k)t)^{-n} (1-e(k')t)^{-\ell} d_k^{(n)} d_{k'}^{(\ell)} \mathcal{D}_K(x^{(\alpha)})^p t^{n+\ell} + \mathcal{D}_K(x^{(\alpha)})^p \otimes 1 \\
&= \mathcal{D}_K(x^{(\alpha)})^p \otimes 1 + \sum_{n, \ell=0}^{p-1} (-1)^{n+\ell} h(k')^{(\ell)} h(k)^{(n)} \otimes (1-e(k)t)^{-n} (1-e(k')t)^{-\ell} \cdot \\
&\quad \cdot \left( \delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\alpha)})^p - \delta_{\ell 0} \delta_{n 1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) e(k) - \delta_{\ell 1} \delta_{n 0} (\delta_{\alpha, \epsilon_{k'} + \epsilon_{-k'}} + \delta_{\alpha, \epsilon_0}) e(k') \right) t^{n+\ell} \\
&= \mathcal{D}_K(x^{(\alpha)})^p \otimes 1 + 1 \otimes \mathcal{D}_K(x^{(\alpha)})^p + h(k)^{(1)} \otimes (1-e(k)t)^{-1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) e(k)t \\
&\quad + h(k')^{(1)} \otimes (1-e(k')t)^{-1} (\delta_{\alpha, \epsilon_{k'} + \epsilon_{-k'}} + \delta_{\alpha, \epsilon_0}) e(k')t \\
&= \mathcal{D}_K(x^{(\alpha)})^p \otimes 1 + 1 \otimes \mathcal{D}_K(x^{(\alpha)})^p \\
&\quad + \delta_{\alpha, \epsilon_i + \epsilon_{-i}} \delta_{ik} h(k)^{(1)} \otimes (1-e(k)t)^{-1} e(k)t + \delta_{\alpha, \epsilon_0} h(k)^{(1)} \otimes (1-e(k)t)^{-1} e(k)t \\
&\quad + \delta_{\alpha, \epsilon_i + \epsilon_{-i}} \delta_{ik'} h(k')^{(1)} \otimes (1-e(k')t)^{-1} e(k')t + \delta_{\alpha, \epsilon_0} h(k')^{(1)} \otimes (1-e(k')t)^{-1} e(k')t.
\end{aligned}$$

Hence, when  $\alpha \neq \epsilon_i + \epsilon_{-i}$ ,  $\epsilon_0$ , we get

$$\begin{aligned}
\Delta((\mathcal{D}_K(x^{(\alpha)}))^p) &\equiv (\mathcal{D}_K(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (\mathcal{D}_K(x^{(\alpha)}))^p \\
&\in I_{t,q} \otimes U_{t,q}(\mathbf{K}(2n+1; \underline{1})) + U_{t,q}(\mathbf{K}(2n+1; \underline{1})) \otimes I_{t,q}.
\end{aligned}$$

And when  $\alpha = \epsilon_i + \epsilon_{-i}$ ,  $1 \leq i \leq n$ ,

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) &= \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) \otimes 1 + \sum_{n, \ell=0}^{p-1} (-1)^{n+\ell} h(k')^{(\ell)} h(k)^{(n)} \otimes (1-e(k')t)^{-\ell} (1-e(k)t)^{-n} \\
&\quad \times \left( \delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{\ell 0} \delta_{n 1} \delta_{ik} e(k) - \delta_{\ell 1} \delta_{n 0} \delta_{ik'} e(k') \right) t^{n+\ell} \\
&= \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) \otimes 1 + 1 \otimes \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) + \delta_{ik} h(k)^{(1)} \otimes (1-e(k)t)^{-1} e(k)t \\
&\quad + \delta_{ik'} h(k')^{(1)} \otimes (1-e(k')t)^{-1} e(k')t.
\end{aligned}$$

Combing this with (3.23), we obtain

$$\begin{aligned}
&\Delta(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) \\
&= (\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) \otimes 1 + 1 \otimes (\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) \\
&\in I_{t,q} \otimes U_{t,q}(\mathbf{K}(2n+1; \underline{1})) + U_{t,q}(\mathbf{K}(2n+1; \underline{1})) \otimes I_{t,q}.
\end{aligned}$$

When  $\alpha = \epsilon_0$ ,

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^{(\epsilon_0)})) &= \mathcal{D}_K(x^{(\epsilon_0)}) \otimes 1 + \sum_{n, \ell=0}^{p-1} (-1)^{n+\ell} h(k')^{(\ell)} h(k)^{(n)} \otimes (1-e(k')t)^{-\ell} (1-e(k)t)^{-n} \\
&\quad \times \left( \delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\epsilon_0)}) - \delta_{\ell 0} \delta_{n 1} e(k) - \delta_{n 0} \delta_{\ell 1} e(k') \right) t^{n+\ell}
\end{aligned}$$

$$= \mathcal{D}_K(x^{(\epsilon_0)}) \otimes 1 + 1 \otimes \mathcal{D}_K(x^{(\epsilon_0)}) + h(k)^{(1)} \otimes (1 - e(k)t)^{-1} e(k)t + h(k')^{(1)} \otimes (1 - e(k')t)^{-1} e(k')t.$$

Combing this with (3.23), we obtain

$$\begin{aligned} \Delta(\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) &= (\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) \otimes 1 + 1 \otimes (\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) \\ &\in I_{t,q} \otimes U_{t,q}(\mathbf{K}(2n+1; \underline{1})) + U_{t,q}(\mathbf{K}(2n+1; \underline{1})) \otimes I_{t,q}. \end{aligned}$$

Thus, we show the ideal  $I_{t,q}$  is also a coideal of the Hopf algebra  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ .

(II) By Lemmas 3.8, 3.14 & 3.15, we have

$$\begin{aligned} S(\mathcal{D}_K(x^{(\alpha)})^p) &= (-1)^p (1 - e(k')t)^{-p(\alpha_{k'} - \alpha_{-k'})} (1 - e(k)t)^{-p(\alpha_k - \alpha_{-k})} \times \\ &\quad \times \sum_{n, \ell=0}^{\infty} (\delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\alpha)})^p - \delta_{\ell 0} \delta_{n 1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) e(k) \\ &\quad - \delta_{\ell 1} \delta_{n 0} (\delta_{\alpha, \epsilon_{k'} + \epsilon_{-k'}} + \delta_{\alpha, \epsilon_0}) e(k')) h(k)_1^{(n)} h(k')_1^{(\ell)} t^{n+\ell} \\ &\equiv -\mathcal{D}_K(x^{(\alpha)})^p + (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) e(k) h(k)_1^{(1)} t \\ &\quad + (\delta_{\alpha, \epsilon_{k'} + \epsilon_{-k'}} + \delta_{\alpha, \epsilon_0}) e(k') h(k')_1^{(1)} t. \end{aligned} \tag{3.24}$$

Thus, when  $\alpha \neq \epsilon_i + \epsilon_{-i}, \epsilon_0$ , for  $1 \leq i \leq n$ , we have  $S(\mathcal{D}_K(x^{(\alpha)})^p) \equiv -\mathcal{D}_K(x^{(\alpha)})^p \in I_{t,q}$ .

When  $\alpha = \epsilon_i + \epsilon_{-i}$ , for  $1 \leq i \leq n$ ,

$$\begin{aligned} S(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) &= - \sum_{n, \ell=0}^{\infty} d_k^{(n)} d_{k'}^{(\ell)} (\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) h(k)_1^{(n)} h(k')_1^{(\ell)} t^{n+\ell} \\ &= - \sum_{n, \ell=0}^{\infty} (\delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{\ell 0} \delta_{n 1} \delta_{ik} e(k) - \delta_{\ell 1} \delta_{n 0} \delta_{ik'} e(k')) h(k')_1^{(\ell)} h(k)_1^{(n)} t^{n+\ell} \\ &= -\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) + \delta_{ik} e(k) h(k)_1^{(1)} t + \delta_{ik'} e(k') h(k')_1^{(1)} t. \end{aligned}$$

Combing this with (3.24), we have

$$S(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) = -(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) \in I_{t,q}.$$

For  $\alpha = \epsilon_0$ ,  $S(\mathcal{D}_K(x^{(\epsilon_0)})) = -\mathcal{D}_K(x^{(\epsilon_0)}) + e(k)h(k)_1^{(1)}t + e(k')h(k')_1^{(1)}t$ , so

$$S(\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) = -(\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) \in I_{t,q}.$$

Hence, we proved that the ideal  $I_{t,q}$  is preserved by the antipode  $S$  of the quantization  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ .

(III) It is obvious to notice that  $\varepsilon(\mathcal{D}_K(x^{(\alpha)})) = 0$ , for  $0 \leq \alpha \leq \tau$ .

This completes the proof.  $\square$

#### 4. Quantization of horizontal type for Lie bialgebra of Cartan type $\mathbf{K}$

In this section, we assume that  $n \geq 2$ . Take  $h := \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$  and  $e := \mathcal{D}_K(x^{\epsilon_k + \epsilon_m})$ , ( $1 \leq k \neq |m| \leq n$ ) and denote by  $\mathcal{F}(k, m)$  the corresponding Drinfel'd twist. Set  $d^{(\ell)} = \frac{1}{\ell!}(\text{ad } e)^\ell$ .

For  $m \in \{-1, \dots, -n, 1, \dots, n\}$ , set  $\sigma(m) := \begin{cases} -1, & m > 0, \\ 1, & m < 0. \end{cases}$  Using the horizontal Drinfel'd

twists  $\mathcal{F}(k, m)$  and the same discussion in Sections 2, 3, we obtain some new quantizations of horizontal type for the universal enveloping algebra of the special algebra  $\mathbf{K}(2n+1; \underline{1})$ .

To simplify the formulas, let us introduce the operator  $d^{(\ell)}$  on  $U(\mathbf{K})$  defined by  $d^{(\ell)} = \frac{1}{\ell!}(\text{ad } e)^\ell$ . Use induction on  $\ell$ , we can get

$$d^{(\ell)}(\mathcal{D}_K(x^\alpha)) = \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+\ell(2\epsilon_k+\epsilon_{-k})-j(\epsilon_k+\epsilon_{-k})-(\ell-j)\epsilon_0}),$$

where  $A_j = (-1)^j \frac{1}{j!} \prod_{i=0}^{j-1} (\alpha_{-k} - j)$ ,  $B_{\ell-j} = \sigma(m)^{\ell-j} \frac{1}{(\ell-j)!} \prod_{i=0}^{\ell-j-1} (\alpha_{-m} - i)$ .

Recall the vertical basic twist of Cartan type  $\mathbf{H}$  Lie algebra in [28] is given by For  $h := D_H(x^{\epsilon_k+\epsilon_{-k}})$  and  $e := D_H(x^{\epsilon_k+\epsilon_m})$ , ( $1 \leq k, |m| \leq n$ ,  $m \neq \pm k$ ), (and the twists of the Hamiltonian algebra in characteristic  $p > 0$  is given by  $h = D_H(x^{\epsilon_k+\epsilon_{-k}})$ ,  $e = 2D_H(x^{\epsilon_k+\epsilon_m})$  ( $1 \leq k, |m| \leq n$ ,  $m \neq \pm k$ )) using the quantizations of Cartan type  $\mathbf{H}$  Lie algebra and the formulas  $\mathcal{D}_K(x^\alpha) = (2 - \sum_{i=1}^n (\alpha_i + \alpha_{-i})) x^{\alpha-\epsilon_0} \partial_0 + \sum_{i=1}^n \alpha_0 x^{\alpha-\epsilon_0} (\partial_i + \partial_{-i}) + D_H(x^\alpha)$ . We have the following theorem which gives the quantization of  $U(\mathbf{K})$  by Drinfel'd twist  $\mathcal{F}(k, m)$ .

LEMMA 4.1. *Fix two distinguished elements  $h := \mathcal{D}_K(x^{\epsilon_k+\epsilon_{-k}})$  and  $e := \mathcal{D}_K(x^{\epsilon_k+\epsilon_m})$ , ( $1 \leq k \neq |m| \leq n$ ), the corresponding horizontal quantization of  $U(\mathbf{K}_{\mathbb{Z}}^+)$  over  $U(\mathbf{K}_{\mathbb{Z}}^+)[[t]]$  by Drinfel'd twist  $\mathcal{F}(k, m)$  with the product undeformed is given by*

$$(4.1) \quad \Delta(\mathcal{D}_K(x^\alpha)) = \mathcal{D}_K(x^\alpha) \otimes (1-et)^{\alpha_k-\alpha_{-k}} + \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} (-1)^\ell A_j B_{\ell-j} h^{(\ell)} \otimes (1-et)^{-\ell} \times \\ \times \mathcal{D}_K(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) t^\ell,$$

$$(4.2) \quad S(\mathcal{D}_K(x^\alpha)) = -(1-et)^{\alpha_k-\alpha_{-k}} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) h_1^{(\ell)} t^\ell,$$

$$(4.3) \quad \varepsilon(\mathcal{D}_K(x^\alpha)) = 0,$$

where  $A_j = (-1)^j \frac{1}{j!} \prod_{i=0}^{j-1} (\alpha_{-k} - j)$ ,  $B_{\ell-j} = \sigma(m)^{\ell-j} \frac{1}{(\ell-j)!} \prod_{i=0}^{\ell-j-1} (\alpha_{-m} - i)$ ,  $A_0 = B_0 = 1$ .

We firstly make the modulo  $p$  reduction for the quantizations of  $U(\mathbf{K}_{\mathbb{Z}}^+)$  in Lemma 4.1 to yield the horizontal quantizations of  $U(\mathbf{K}(2n+1; \underline{1}))$  over  $U_t(\mathbf{K}(2n+1; \underline{1}))$ .

THEOREM 4.2. *Fix distinguished elements  $h = \mathcal{D}_K(x^{\epsilon_k+\epsilon_{-k}})$ ,  $e = \mathcal{D}_K(x^{\epsilon_k+\epsilon_m})$  ( $1 \leq |m| \neq k \leq n$ ), the corresponding horizontal quantization of  $U(\mathbf{K}(2n+1; \underline{1}))$  over  $U_t(\mathbf{K}(2n+1; \underline{1}))$  with the product undeformed is given by*

$$(4.4) \quad \Delta(\mathcal{D}_K(x^{(\alpha)})) = \mathcal{D}_K(x^{(\alpha)}) \otimes (1-et)^{\alpha_k-\alpha_{-k}} + \\ \sum_{\ell=0}^{p-1} \sum_{j=0}^{\ell} (-1)^\ell \bar{A}_j \bar{B}_{\ell-j} h^{(\ell)} \otimes (1-et)^{-\ell} \mathcal{D}_K(x^{(\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k}))}) t^\ell,$$

$$(4.5) \quad S(\mathcal{D}_K(x^{(\alpha)})) = -(1-et)^{\alpha_k-\alpha_{-k}} \sum_{\ell=0}^{p-1} \sum_{j=0}^{\ell} \bar{A}_j \bar{B}_{\ell-j} \mathcal{D}_K(x^{(\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k}))}) t^\ell,$$

$$(4.6) \quad \varepsilon(\mathcal{D}_K(x^{(\alpha)})) = 0,$$

where  $0 \leq \alpha \leq \tau$  for  $2n+4 \not\equiv 0 \pmod{p}$  and  $0 \leq \alpha < \tau$ , for  $2n+4 \equiv 0 \pmod{p}$ . And  $\bar{A}_j \equiv (-1)^j \binom{\alpha_m+j}{j} \pmod{p}$ , for  $0 \leq j \leq \alpha_{-k}$ ,  $B_{\ell-j} \equiv \sigma(m)^{\ell-j} \binom{\alpha_k+\ell-j}{\ell-j} \pmod{p}$  for  $0 \leq \ell-j \leq \alpha_{-m}$ , otherwise,  $\bar{A}_j = \bar{B}_{\ell-j} = 0$ .

To describe  $\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1}))$  explicitly, we still need an auxiliary Lemma.

LEMMA 4.3. Set  $e = \mathcal{D}_K(x^{\epsilon_k+\epsilon_m})$ ,  $d^{(\ell)} = \frac{1}{\ell!} \text{ad } e$ . Then

- (i)  $d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) = \sum_{j=0}^{\ell} \bar{A}_j \bar{B}_{\ell-j} \mathcal{D}_K(x^{(\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k}))})$ ,
- (ii)  $d^{(\ell)}(\mathcal{D}_K(x^{\epsilon_i+\epsilon_{-i}})) = \delta_{\ell 0} \mathcal{D}_K(x^{\epsilon_i+\epsilon_{-i}}) + \delta_{\ell 1} (\delta_{i,-m} - \delta_{im} - \delta_{ik}) e$ , for  $1 \leq i \leq n$ ,  
 $d^{(\ell)}(\mathcal{D}_K(x^{\epsilon_0})) = \delta_{\ell 0} \mathcal{D}_K(x^{\epsilon_0})$ .
- (iii)  $d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})^p) = \delta_{\ell 0} \mathcal{D}_K(x^{(\alpha)})^p - \delta_{\ell 1} (\delta_{\alpha, \epsilon_k+\epsilon_{-k}} - \sigma(m) \delta_{\alpha, \epsilon_m+\epsilon_{-m}}) e$ .

Based on Theorem 4.2 and Lemma 4.3, we arrive at

THEOREM 4.4. Fix distinguished elements  $h := \mathcal{D}_K(x^{\epsilon_k+\epsilon_{-k}})$ ,  $e := \mathcal{D}_K(x^{\epsilon_k+\epsilon_m})$ , with  $1 \leq k \neq |m| \leq n$ ; there exists a noncommutative and noncocommutative Hopf algebra (of horizontal type)  $(\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})), m, \iota, \Delta, S, \varepsilon)$  over  $\mathcal{K}[t]_p^{(q)}$  with the product undeformed, whose coalgebra structure is given by

$$(4.7) \quad \Delta(\mathcal{D}_K(x^{(\alpha)})) = \mathcal{D}_K(x^{(\alpha)}) \otimes (1-et)^{\alpha_k-\alpha_{-k}} + \sum_{\ell=0}^{p-1} (-1)^{\ell} h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) t^{\ell},$$

$$(4.8) \quad S(\mathcal{D}_K(x^{(\alpha)})) = -(1-et)^{\alpha_{-k}-\alpha_k} \sum_{\ell=0}^{p-1} d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) h_1^{(\ell)} t^{\ell},$$

$$(4.9) \quad \varepsilon(\mathcal{D}_K(x^{(\alpha)})) = 0,$$

where  $0 \leq \alpha \leq \tau$ ,  $\dim_{\mathcal{K}} \mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})) = p^{2n+1+1}$  in case of  $2n+4 \not\equiv 0 \pmod{p}$ , and  $0 \leq \alpha < \tau$ ,  $\dim_{\mathcal{K}} \mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})) = p^{2n+1}$  otherwise.

## 5. Other Drinfel'd twists and some Open Questions

Besides the Drinfel'd twists (i), (ii) we interested in Section 2, there are another two kinds of interesting Drinfel'd twists:

$$(iii) \quad h = \mathcal{D}_K(x^{\epsilon_k+\epsilon_{-k}}), \quad e = \mathcal{D}_K(x^{\epsilon_k+\epsilon_0}) \quad (1 \leq k \leq n);$$

$$(ix) \quad h = \mathcal{D}_K(x^{\epsilon_0}), \quad e = \mathcal{D}_K(x^{\epsilon_k+\epsilon_0}) \quad (1 \leq |k| \leq n).$$

We can see that the elements in (iii) and (ix) also satisfies  $[h, e] = e$ , so we can get triangular Lie bialgebra structure on  $\mathcal{K}$  given by the classical Yang-Baxter  $r$ -matrix  $r := h \otimes e - e \otimes h$ . We can also get some new Drinfel'd twists. We still use  $\mathcal{F}(k)$ 's to denote these Drinfel'd twists, and call them the vertical twists due to the degree  $\|\epsilon_k + \epsilon_0\| = 1$ .

The basic theory is given in Sections 3 and 4, here we only list some main calculations.

**5.1. Construction of Drinfel'd twist.** To simplify the formulas, let us introduce the operator  $d^{(\ell)}$  on  $U(\mathbf{K})$  defined by  $d^{(\ell)} = \frac{1}{\ell!}(\text{ad } e)^\ell$ . Then we can get

LEMMA 5.1. *For  $\mathcal{D}_K(x^\alpha) \in U(\mathbf{K})$ , the following equalities hold*

$$(5.1) \quad d^{(\ell)}(\mathcal{D}_K(x^\alpha)) = \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}),$$

$$(5.2) \quad \mathcal{D}_K(x^\alpha) \cdot e^m = \sum_{\ell=0}^m (-1)^\ell \binom{m}{\ell} \ell! \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}),$$

$$(5.3) \quad (\text{ad } \mathcal{D}_K(x^\alpha))^\ell(e) = \sum_{j=0}^{\ell} \binom{\ell}{j} \prod_{i=0}^{\ell-j-1} ((i-1)\|\alpha\| + \alpha_0) \prod_{i=0}^{j-1} (i\alpha_k - (i-1)\alpha_{-k}) \cdot \mathcal{D}_K(x^{\ell\alpha+\epsilon_k-j(\epsilon_k+\epsilon_{-k})-(\ell-j-1)\epsilon_0}),$$

where  $A_j = \frac{1}{j!} \prod_{i=0}^{j-1} (i - \alpha_{-k})$ ,  $B_{\ell-j} = \frac{1}{(\ell-j)!} \prod_{i=0}^{\ell-j-1} (\|\alpha\| - \alpha_0 + i)$ ,  $A_0 = B_0 = 1$ .

PROOF. For (5.1), use induction on  $\ell$ . When  $\ell = 1$ ,

$$\begin{aligned} d(\mathcal{D}_K(x^\alpha)) &= [\mathcal{D}_K(x^{\epsilon_0+\epsilon_k}), \mathcal{D}_K(x^\alpha)] \\ &= (\alpha_0 - (2 - \sum_{i=1}^n (\alpha_i + \alpha_{-i}))) \mathcal{D}_K(x^{\alpha+\epsilon_k}) - \alpha_{-k} \mathcal{D}_K(x^{\alpha+\epsilon_0-\epsilon_{-k}}) \\ &= (\|\alpha\| - \alpha_0) \mathcal{D}_K(x^{\alpha+\epsilon_k}) - \alpha_{-k} \mathcal{D}_K(x^{\alpha+\epsilon_0-\epsilon_{-k}}). \end{aligned}$$

When  $\ell \geq 1$ , we have

$$\begin{aligned} d^{(\ell+1)}(\mathcal{D}_K(x^\alpha)) &= \frac{d}{\ell+1} d^{(\ell)}(\mathcal{D}_K(x^\alpha)) = \frac{1}{\ell+1} \sum_{j=0}^{\ell} A_j B_{\ell-j} d(\mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})})) \\ &= \frac{1}{\ell+1} \sum_{j=0}^{\ell} A_j B_{\ell-j} \left( (\alpha_0 + j) - (2 - \sum_{i=1}^n \alpha_i + \alpha_{-i} + \ell - j - j) \right) \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})+\epsilon_k}) \\ &\quad - (\alpha_{-k} - j) \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})+\epsilon_0-\epsilon_{-k}}) \\ &= \frac{1}{\ell+1} \sum_{j=0}^{\ell} A_j B_{\ell-j} \left( (\|\alpha\| - \alpha_0 + \ell - j) \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})+\epsilon_k}) + \right. \\ &\quad \left. + (j - \alpha_{-k}) \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+(j+1)(\epsilon_0-\epsilon_{-k})}) \right) \\ &= \frac{1}{\ell+1} \sum_{j=0}^{\ell} (\ell - j + 1) A_j B_{\ell-j+1} \mathcal{D}_K(x^{\alpha+(\ell-j+1)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}) \\ &\quad + \frac{1}{\ell+1} \sum_{j=0}^{\ell} (j+1) A_{j+1} B_{\ell-j} \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+(j+1)(\epsilon_0-\epsilon_{-k})}) \\ &= \sum_{j=0}^{\ell+1} A_j B_{\ell+1-j} \mathcal{D}_K(x^{\alpha+(\ell+1-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}). \end{aligned}$$

(5.2) follows from (5.1).

For (5.3), use induction on  $\ell$ . When  $\ell = 1$ , we have

$$\begin{aligned} \text{ad } \mathcal{D}_K(x^\alpha)(e) &= [\mathcal{D}_K(x^\alpha), \mathcal{D}_K(x^{\epsilon_k + \epsilon_0})] \\ &= (2 - \sum_{i=1}^n (\alpha_i + \alpha_{-i}) - \alpha_0) \mathcal{D}_K(x^{\alpha + \epsilon_k}) + \alpha_{-k} \mathcal{D}_K(x^{\alpha + \epsilon_0 - \epsilon_{-k}}) \\ &= (-\|\alpha\| + \alpha_0) \mathcal{D}_K(x^{\alpha + \epsilon_k}) + \alpha_{-k} \mathcal{D}_K(x^{\alpha + \epsilon_0 - \epsilon_{-k}}). \end{aligned}$$

When  $\ell \geq 1$ , we have

$$\begin{aligned} &(\text{ad } \mathcal{D}_K(x^\alpha))^{\ell+1}(e) \\ &= \text{ad } \mathcal{D}_K(x^\alpha) \sum_{j=0}^{\ell} \binom{\ell}{j} \prod_{i=0}^{\ell-j-1} ((i-1)\|\alpha\| + \alpha_0) \prod_{i=0}^{j-1} (i\alpha_k - (i-1)\alpha_{-k}) \mathcal{D}_K(x^{\ell\alpha + \epsilon_k - j(\epsilon_k + \epsilon_{-k}) - (\ell-j-1)\epsilon_0}) \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} \prod_{i=0}^{\ell-j} ((i-1)\|\alpha\| + \alpha_0) \prod_{i=0}^{j-1} (i\alpha_k - (i-1)\alpha_{-k}) \mathcal{D}_K(x^{(\ell+1)\alpha + \epsilon_k - (\ell-j)\epsilon_0 - j(\epsilon_k + \epsilon_{-k})}) \\ &\quad + \sum_{j=0}^{\ell} \binom{\ell}{j} \prod_{i=0}^{\ell-j-1} ((i-1)\|\alpha\| + \alpha_0) \prod_{i=0}^j (i\alpha_k - (i-1)\alpha_{-k}) \mathcal{D}_K(x^{(\ell+1)\alpha + \epsilon_k - (\ell-j-1)\epsilon_0 - (j+1)(\epsilon_k + \epsilon_{-k})}) \\ &= \sum_{j=0}^{\ell+1} \binom{\ell+1}{j} \prod_{i=0}^{\ell-j} ((i-1)\|\alpha\| + \alpha_0) \prod_{i=0}^{j-1} (i\alpha_k - (i-1)\alpha_{-k}) \mathcal{D}_K(x^{(\ell+1)\alpha + \epsilon_k - (\ell-j)\epsilon_0 - j(\epsilon_k + \epsilon_{-k})}). \end{aligned}$$

Thus we can get (5.3).  $\square$

As the proof in Section 3, we can also get

LEMMA 5.2. For  $a \in \mathbb{F}$ ,  $\alpha \in \mathbb{Z}^{2n+1}$ , and  $\mathcal{D}_K(x^\alpha) \in \mathbf{K}$ , the following equalities hold

$$(5.4) \quad ((\mathcal{D}_K(x^\alpha))^s \otimes 1) \cdot F_a = F_{a+s(\alpha_k - \alpha_{-k})} \cdot ((\mathcal{D}_K(x^\alpha))^s \otimes 1),$$

$$(5.5) \quad (\mathcal{D}_K(x^\alpha))^s \cdot u_a = u_{a+s(\alpha_k - \alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}((\mathcal{D}_K(x^\alpha))^s) h_{1-a}^{(\ell)} t^\ell,$$

$$(5.6) \quad (1 \otimes (\mathcal{D}_K(x^\alpha))^s) \cdot F_a = \sum_{\ell=0}^{\infty} (-1)^\ell F_{a+\ell} (h_a^{(\ell)} \otimes d^{(\ell)}(\mathcal{D}_K(x^\alpha)^s)) t^\ell.$$

LEMMA 5.3. For  $s \geq 1$ , we have

$$(5.7) \quad \Delta((\mathcal{D}_K(x^\alpha))^s) = \sum_{\substack{0 \leq j \leq s \\ \ell \geq 0}} (-1)^\ell \binom{s}{j} (\mathcal{D}_K(x^\alpha))^j h^{(\ell)} \otimes (1-et)^{j(\alpha_k - \alpha_{-k}) - \ell} (d^{(\ell)}(\mathcal{D}_K(x^\alpha))^{s-j}) t^\ell,$$

$$(5.8) \quad S((\mathcal{D}_K(x^\alpha))^s) = (-1)^s (1-et)^{-s(\alpha_k - \alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}(\mathcal{D}_K(x^\alpha)^s) h_1^{(\ell)} t^\ell.$$

LEMMA 5.4. Fix two distinguished elements  $h := \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$  and  $e := \mathcal{D}_K(x^{\epsilon_k + \epsilon_0})$ , ( $1 \leq k \leq n$ , the corresponding vertical quantization of  $U(\mathbf{K}_{\mathbb{Z}}^+)$  over  $U(\mathbf{K}_{\mathbb{Z}}^+)[[t]]$  by Drinfel'd twist  $\mathcal{F}(k, 0)$  with the product undeformed is given by

$$(5.9) \quad \Delta(\mathcal{D}_K(x^\alpha)) = \mathcal{D}_K(x^\alpha) \otimes (1-et)^{\alpha_k - \alpha_{-k}}$$

$$\begin{aligned}
& + \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} (-1)^{\ell} A_j B_{\ell-j} h^{(\ell)} \otimes (1-et)^{-\ell} \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}) t^{\ell}, \\
(5.10) \quad S(\mathcal{D}_K(x^{\alpha})) &= -(1-et)^{\alpha-k-\alpha_k} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} A_j B_{\ell-j} \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}) h_1^{(\ell)} t^{\ell}, \\
(5.11) \quad \varepsilon(\mathcal{D}_K(x^{\alpha})) &= 0,
\end{aligned}$$

where  $A_j, B_{\ell-j}$  as defined in 5.1.

Firstly, we make the modulo  $p$  reduction for the quantizations of  $U(\mathbf{K}_2^+)$  in Lemma 5.4 to yield the vertical quantizations of  $U(\mathbf{K}(2n+1; \underline{1}))$  over  $U_t(\mathbf{K}(2n+1; \underline{1}))$ .

**THEOREM 5.5.** *Fix distinguished elements  $h = \mathcal{D}_K(x^{(\epsilon_k+\epsilon_{-k})})$ ,  $e = \mathcal{D}_K(x^{(\epsilon_k+\epsilon_0)})$  ( $1 \leq k \leq n$ ), the corresponding vertical quantization of  $U(\mathbf{K}(2n+1; \underline{1}))$  over  $U_t(\mathbf{K}(2n+1; \underline{1}))$  with the product undeformed is given by*

$$\begin{aligned}
(5.12) \quad \Delta(\mathcal{D}_K(x^{(\alpha)})) &= \mathcal{D}_K(x^{(\alpha)}) \otimes (1-et)^{\alpha-k-\alpha_k} + \\
& \sum_{\ell=0}^{p-1} \sum_{j=0}^{\ell} (-1)^{\ell} \bar{A}_j \bar{B}_{\ell-j} h^{(\ell)} \otimes (1-et)^{-\ell} \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}) t^{\ell}, \\
(5.13) \quad S(\mathcal{D}_K(x^{(\alpha)})) &= -(1-et)^{\alpha-k-\alpha_k} \sum_{\ell=0}^{p-1} \sum_{j=0}^{\ell} \bar{A}_j \bar{B}_{\ell-j} \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}) h_1^{(\ell)} t^{\ell}, \\
(5.14) \quad \varepsilon(\mathcal{D}_K(x^{(\alpha)})) &= 0,
\end{aligned}$$

where  $0 \leq \alpha \leq \tau$ , if  $2n+4 \not\equiv 0 \pmod{p}$ ;  $0 \leq \alpha < \tau$ , if  $2n+4 \equiv 0 \pmod{p}$ . And  $\bar{A}_j \equiv (-1)^j \binom{\alpha_0+j}{j} \pmod{p}$ , for  $0 \leq j \leq \alpha_{-k}$ . Otherwise,  $\bar{A}_j = 0$ ,  $B_{\ell-j} \equiv (\ell-j)! \binom{\alpha_k+\ell-j}{j} B_{\ell-j} \pmod{p}$ .

**PROOF.** As the same argument of Theorem 4.2, we can prove this theorem once we notice that

$$\begin{aligned}
& \frac{A_j B_{\ell-j} (\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k}))!}{\alpha!} \\
&= \frac{\prod_{i=0}^{j-1} (i-\alpha_{-k})}{j!} \frac{\prod_{i=0}^{\ell-j-1} (\|\alpha\|-\alpha_0+j)}{(\ell-j)!} \frac{(\alpha_k+(\ell-j))! (\alpha_{-k}-j)! (\alpha_0+j)!}{\alpha_k! \alpha_{-k}! \alpha_0!} \\
&= (-1)^j \frac{\alpha_{-k}(\alpha_{-k}-1) \cdots (\alpha_{-k}-j+1) (\alpha_{-k}-j)! (\alpha_0+j)! (\alpha_k+(\ell-j))!}{\alpha_{-k}! \alpha_0! j! \alpha_k!} B_{\ell-j} \\
&= \bar{A}_j \binom{\alpha_k+\ell-j}{\ell-j} (\ell-j)! B_{\ell-j} = \bar{A}_j \bar{B}_{\ell-j}.
\end{aligned}$$

This completes the proof.  $\square$

Here we also need the following auxiliary Lemma

**LEMMA 5.6.** *Set  $e = \mathcal{D}_K(x^{(\epsilon_k+\epsilon_0)})$ ,  $d^{(\ell)} = \frac{1}{\ell!} \text{ad } e$ . Then*

$$(i) \quad d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) = \sum_{j=0}^{\ell} \bar{A}_j \bar{B}_{\ell-j} \mathcal{D}_K(x^{\alpha+(\ell-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}),$$



- (ii)  $d^{(\ell)}(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) = \delta_{\ell 0} \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{\ell 1} \delta_{i,k} e,$   
 $d^{(\ell)}(\mathcal{D}_K(x^{(\epsilon_0)})) = \delta_{\ell 0} \mathcal{D}_K(x^{(\epsilon_0)}) - \delta_{\ell 1} e,$
- (iii)  $d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})^p) = \delta_{\ell 0} (\mathcal{D}_K(x^{(\alpha)})^p) - \delta_{\ell 1} (\delta_{\alpha, \epsilon_0} + \delta_{\alpha, \epsilon_k + \epsilon_{-k}}) e.$

PROOF. (i) follows from the proof of Lemma 5.1.

(ii) We can see that  $\bar{A}_0 = \bar{B}_0 = 1.$

For  $\alpha = \epsilon_i + \epsilon_{-i}$ ,  $\bar{A}_1 = \delta_{ik}(-1) \binom{\alpha_0+1}{1} = -\delta_{ik}$ ;  $\bar{B}_1 = \binom{\alpha_k+1}{1} B_1 = (\alpha_k + 1)(\|\epsilon_i + \epsilon_{-i}\|) = 0.$

For  $\alpha = \epsilon_0$ ,  $\bar{A}_1 = 0$ ,  $\bar{B}_1 = 1! \binom{\alpha_0+1}{1} (\|\epsilon_0\| - 1) = -1.$  Then we can get (ii) by (i).

(iii) By (3.7) and Lemma 5.1, we have

$$\begin{aligned}
d(\mathcal{D}_K(x^{(\alpha)})^p) &\equiv (-1)^p (\text{ad } \mathcal{D}_K(x^{(\alpha)})^p)(e) \\
&= (-1) \frac{1}{(\alpha!)^p} \sum_{j=0}^p \binom{p}{j} \prod_{i=0}^{p-j-1} ((i-1)\|\alpha\| + \alpha_0) \prod_{i=0}^{j-1} (i\alpha_k - (i-1)\alpha_{-k}) \mathcal{D}_K(x^{p\alpha + \epsilon_k - (p-j-1)\epsilon_0 - p(\epsilon_k + \epsilon_{-k})}) \\
&\equiv -\frac{1}{\alpha!} \prod_{i=0}^{p-1} ((i-1)\|\alpha\| + \alpha_0) \mathcal{D}_K(x^{p\alpha + \epsilon_k + \epsilon_0 - p\epsilon_0}) \\
&\quad - \frac{1}{\alpha!} \prod_{i=0}^{p-1} (i\alpha_k - (i-1)\alpha_{-k}) \mathcal{D}_K(x^{p\alpha + \epsilon_k + \epsilon_0 - p(\epsilon_k + \epsilon_{-k})}) \pmod{p} \\
&\equiv -\delta_{\alpha, \epsilon_0} e - \delta_{\alpha, \epsilon_k + \epsilon_{-k}} e \pmod{p, J}.
\end{aligned}$$

This completes the proof.  $\square$

**THEOREM 5.7.** Fix distinguished elements  $h := \mathcal{D}_K(x^{(\epsilon_k + \epsilon_{-k})})$ ,  $e := \mathcal{D}_K(x^{(\epsilon_k + \epsilon_0)})$ , with  $1 \leq k \leq n$ , there is a noncommutative and noncocommutative Hopf algebra (of vertical type)  $(\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})), m, \iota, \Delta, S, \varepsilon)$  over  $\mathcal{K}[t]_p^{(q)}$  with the product undeformed, whose coalgebra structure is given by

$$(5.15) \quad \Delta(\mathcal{D}_K(x^{(\alpha)})) = \mathcal{D}_K(x^{(\alpha)}) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) t^\ell,$$

$$(5.16) \quad S(\mathcal{D}_K(x^{(\alpha)})) = -(1-et)^{\alpha_k - \alpha_{-k}} \sum_{\ell=0}^{p-1} d^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) h_1^{(\ell)} t^\ell,$$

$$(5.17) \quad \varepsilon(\mathcal{D}_K(x^{(\alpha)})) = 0,$$

for  $0 \leq \alpha \leq \tau$ , which is finite-dimensional with  $\dim_{\mathcal{K}\mathbf{u}_{t,q}}(\mathbf{K}(2n+1; \underline{1})) = p^{p^{2n+1}+1}$ , when  $2n+4 \not\equiv 0 \pmod{p}$ . And  $0 \leq \alpha < \tau$ ,  $\dim_{\mathcal{K}\mathbf{u}_{t,q}}(\mathbf{K}(2n+1; \underline{1})) = p^{p^{2n+1}}$ , when  $2n+4 \equiv 0 \pmod{p}$ .

PROOF. (I) By (5.7) and Lemma 5.6, we have

$$\begin{aligned}
(5.18) \quad \Delta(\mathcal{D}_K(x^{(\alpha)})^p) &= 1 \otimes \mathcal{D}_K(x^{(\alpha)})^p + \mathcal{D}_K(x^{(\alpha)})^p \otimes 1 - h \otimes d(\mathcal{D}_K(x^{(\alpha)})^p) t \\
&= 1 \otimes \mathcal{D}_K(x^{(\alpha)})^p + \mathcal{D}_K(x^{(\alpha)})^p \otimes 1 + h \otimes (1-et)^{-1} (\delta_{\alpha, \epsilon_0} + \delta_{\alpha, \epsilon_i + \epsilon_{-i}} \delta_{ik}) et.
\end{aligned}$$

Thus, when  $\alpha \neq \epsilon_i + \epsilon_{-i}$ ,  $\alpha \neq \epsilon_0$ , we have

$$\begin{aligned} \Delta((\mathcal{D}_K(x^{(\alpha)}))^p) &= 1 \otimes (\mathcal{D}_K(x^{(\alpha)}))^p + (\mathcal{D}_K(x^{(\alpha)}))^p \otimes 1 \\ &\in I_{t,q} \otimes U_{t,q}(\mathbf{K}(2n+1, \underline{1})) + U_{t,q}(\mathbf{K}(2n+1, \underline{1})) \otimes I_{t,q}. \end{aligned}$$

If  $\alpha = \epsilon_i + \epsilon_{-i}$ , we obtain

$$\Delta(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) = \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) \otimes 1 + 1 \otimes \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) + h \otimes (1 - et)^{-1} \delta_{i,k} et.$$

Then

$$\begin{aligned} \Delta(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) &= (\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) \otimes 1 \\ &\quad + 1 \otimes (\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) \\ &\in I_{t,q} \otimes U_{t,q}(\mathbf{K}(2n+1, \underline{1})) + U_{t,q}(\mathbf{K}(2n+1, \underline{1})) \otimes I_{t,q}. \end{aligned}$$

If  $\alpha = \epsilon_0$ , we have

$$\Delta(\mathcal{D}_K(x^{(\epsilon_0)})) = \mathcal{D}_K(x^{(\epsilon_0)}) \otimes 1 + 1 \otimes \mathcal{D}_K(x^{(\epsilon_0)}) + h \otimes (1 - et)^{-1} et.$$

Thus,

$$\begin{aligned} \Delta(\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) &= (\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) \otimes 1 + 1 \otimes (\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) \\ &\in I_{t,q} \otimes U_{t,q}(\mathbf{K}(2n+1, \underline{1})) + U_{t,q}(\mathbf{K}(2n+1, \underline{1})) \otimes I_{t,q}. \end{aligned}$$

Thus, we show the ideal  $I_{t,q}$  is also a coideal of the Hopf algebra  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ .

(II) By (5.8) and Lemma 5.6, we have

$$\begin{aligned} S(\mathcal{D}_K(x^{(\alpha)})^p) &= (-1)\mathcal{D}_K(x^{(\alpha)})^p - (-\delta_{\alpha, \epsilon_k + \epsilon_{-k}} - \delta_{\alpha, \epsilon_0}) e h_1^{(1)} t \\ &= -\mathcal{D}_K(x^{(\alpha)})^p + (\delta_{\alpha, \epsilon_0} + \delta_{\alpha, \epsilon_i + \epsilon_{-i}} \delta_{ik}) e h_1^{(1)} t. \end{aligned}$$

Thus, when  $\alpha \neq \epsilon_i + \epsilon_{-i}$ ,  $\alpha \neq \epsilon_0$ , we have

$$S(\mathcal{D}_K(x^{(\alpha)})^p) = -\mathcal{D}_K(x^{(\alpha)})^p \in I_{t,q}.$$

And then

$$\begin{aligned} S(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) &= -\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p + \delta_{ik} e h_1^{(1)} t + \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{ik} e h_1^{(1)} t \\ &= -(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})^p - \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) \in I_{t,q}, \\ S(\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) &= -\mathcal{D}_K(x^{(\epsilon_0)})^p + e h_1^{(1)} t + \mathcal{D}_K(x^{(\epsilon_0)}) - e h_1^{(1)} t \\ &= -(\mathcal{D}_K(x^{(\epsilon_0)})^p - \mathcal{D}_K(x^{(\epsilon_0)})) \in I_{t,q}. \end{aligned}$$

Therefore, we proved that  $I_{t,q}$  is indeed preserved by antipode  $S$  of  $U_{t,q}(\mathbf{K}(2n+1; \underline{1}))$ .

(III) It is obvious that  $\varepsilon(\mathcal{D}_K(x^{(\alpha)})) = 0$ , for  $0 \leq \alpha \leq \tau$ , or  $0 \leq \alpha < \tau$ .

This completes the proof.  $\square$

**5.2. More quantizations.** In this subsection, the arguments are similar to Section 4, thus we only give some calculation results. Let  $A(k)_j$  and  $A(k')_{j'}$  denote the coefficients of the corresponding quantizations of  $U(\mathbf{K}_{\mathbb{Z}}^+)$  over  $U(\mathbf{K}_{\mathbb{Z}}^+)[[t]]$  given by Drinfel'd twists  $\mathcal{F}(k)$  and  $\mathcal{F}(k')$  as in Corollary 3.6, respectively. Note that  $A(k)_0 = A(k')_0 = 1$ ,  $A(k)_{-1} = A(k')_{-1} = 0$ .

LEMMA 5.8. *Fix distinguished elements  $h(k) = \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$ ,  $e(k) = \mathcal{D}_K(x^{\epsilon_k + \epsilon_0})$  ( $1 \leq k \leq n$ ) and  $h(k') = \mathcal{D}_K(x^{\epsilon_{k'} + \epsilon_{-k'}})$ ,  $e(k') = \mathcal{D}_K(x^{\epsilon_{k'} + \epsilon_0})$  ( $1 \leq k' \leq n$ ) with  $k \neq k'$ . the corresponding quantization of  $U(\mathbf{K}_{\mathbb{Z}}^+)[[t]]$  by the Drinfel'd twist  $\mathcal{F} = \mathcal{F}(k)\mathcal{F}(k')$  with the product undeformed is given by*

$$\begin{aligned} \Delta(\mathcal{D}_K(x^\alpha)) &= \mathcal{D}_K(x^\alpha) \otimes (1-e(k)t)^{\alpha_k - \alpha_{-k}} (1-e(k')t)^{\alpha_{k'} - \alpha_{-k'}} \\ &\quad + \sum_{n,\ell=0}^{\infty} \sum_{j'=0}^{\ell} \sum_{j=0}^n (-1)^{\ell+n} A(k)_j A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell,j'} h(k')^{(\ell)} h(k)^{(n)} \otimes \\ &\quad (1-e(k)t)^{-n} (1-e(k')t)^{-\ell} \mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_k + j'(\epsilon_0 - \epsilon_{-k'}) + (n-j)\epsilon_k + j(\epsilon_0 - \epsilon_{-k})}) t^{n+\ell}, \\ S(\mathcal{D}_K(x^\alpha)) &= -(1-e(k)t)^{\alpha_{-k} - \alpha_k} (1-e(k')t)^{\alpha_{-k'} - \alpha_{k'}} \sum_{n,\ell=0}^{\infty} \sum_{j'=0}^{\ell} \sum_{j=0}^n A(k)_j A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell,j'} \times \\ &\quad \times \mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_{k'} + j'(\epsilon_0 - \epsilon_{-k'}) + (n-j)\epsilon_{k'} + j(\epsilon_0 - \epsilon_{-k})}) h(k)_1^{(n)} h(k')_1^{(\ell)} t^{\ell+n}, \end{aligned}$$

and  $\varepsilon(\mathcal{D}_K(x^\alpha)) = 0$ , where  $C_{n-j}^{\ell,j'} = \frac{1}{(n-j)!} \prod_{i=0}^{n-j-1} (\|\alpha\| - \alpha_0 + (\ell-j') + i)$ , for  $\mathcal{D}_K(x^\alpha) \in \mathbf{K}_{\mathbb{Z}}^+$ .

PROOF. First of all, let us consider

$$\begin{aligned} \Delta(\mathcal{D}_K(x^\alpha)) &= \mathcal{F}(k)\mathcal{F}(k')\Delta_0(\mathcal{D}_K(x^\alpha))\mathcal{F}(k')^{-1}\mathcal{F}(k)^{-1} \\ &= \mathcal{F}(k)\mathcal{D}_K(x^\alpha) \otimes (1-e(k')t)^{\alpha_{k'} - \alpha_{-k'}} F(k) \\ &\quad + \mathcal{F}(k) \left( \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1-e(k')t)^{-\ell} d_{k'}^{(\ell)}(\mathcal{D}_K(x^\alpha)) t^\ell \right) F(k). \end{aligned}$$

By Lemma 2.1 & Corollary 3.6, we can get

$$\begin{aligned} &\mathcal{F}(k)(\mathcal{D}_K(x^\alpha) \otimes (1-e(k')t)^{\alpha_{k'} - \alpha_{-k'}})F(k) \\ &= \mathcal{F}(k)(\mathcal{D}_K(x^\alpha) \otimes 1)(1 \otimes (1-e(k')t)^{\alpha_{k'} - \alpha_{-k'}})F(k) \\ &= \mathcal{F}(k)(\mathcal{D}_K(x^\alpha) \otimes 1)F(k)(1 \otimes (1-e(k')t)^{\alpha_{k'} - \alpha_{-k'}}) \\ &= \mathcal{F}(k)F(k)_{\alpha_k - \alpha_{-k}}(\mathcal{D}_K(x^\alpha) \otimes 1)(1 \otimes (1-e(k')t)^{\alpha_{k'} - \alpha_{-k'}}) \\ &= \mathcal{D}_K(x^\alpha) \otimes (1-e(k)t)^{\alpha_k - \alpha_{-k}} (1-e(k')t)^{\alpha_{k'} - \alpha_{-k'}}, \\ &\mathcal{F}(k) \left( \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1-e(k')t)^{-\ell} d_{k'}^{(\ell)} \mathcal{D}_K(x^\alpha) t^\ell \right) F(k) \\ &= \mathcal{F}(k) \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1-e(k')t)^{-\ell} \sum_{j'=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} \mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_{k'} + j'(\epsilon_0 - \epsilon_{-k'})}) t^\ell F(k) \\ &= \left( \sum_{\ell=0}^{\infty} (-1)^\ell h(k')^{(\ell)} \otimes (1-e(k')t)^{-\ell} \right) \left( \sum_{j'=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} \mathcal{F}(k)(1 \otimes \mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_{k'} + j'(\epsilon_0 - \epsilon_{-k'})})) F(k) t^\ell \right) \end{aligned}$$

$$= \left( \sum_{\ell=0}^{\infty} (-1)^{\ell} h(k')^{\langle \ell \rangle} \otimes (1-e(k')t)^{-\ell} \right) \cdot \left( \sum_{j'=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} \mathcal{F}(k) \sum_{n=0}^{\infty} (-1)^n F(k)_n (h(k)^{\langle n \rangle} \otimes d_k^{(n)}(\mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_k)})) t^n) t^{\ell} \right).$$

Set  $\alpha(\ell, k', j') := \alpha + (\ell - j')\epsilon_{k'} + j'(\epsilon_0 - \epsilon_k)$ . So, it is easy to see

$$\begin{aligned} d_k^{(n)}(\mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_k)})) &= d_k^{(n)}(\mathcal{D}_K(x^{\alpha(\ell, k', j')})) \\ &= \sum_{j=0}^n \frac{\prod_{i=0}^{n-j-1} (\|\alpha(\ell, k', j')\| - \alpha(\ell, k', j')_0 + i)}{(n-j)!} \frac{\prod_{i=0}^{j-1} (i - \alpha(\ell, k', j')_{-k})}{j!} \mathcal{D}_K(x^{\alpha(\ell, k', j') + (n-j)\epsilon_k + j(\epsilon_0 - \epsilon_k)}) \\ &= \sum_{j=0}^n \frac{\prod_{i=0}^{n-j-1} (|\alpha| + (\ell - j') + \alpha_0 + j' - 2 - \alpha_0 - j' + i)}{(n-j)!} A(k)_j \mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_{-k'})+(n-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}) \\ &= \sum_{j=0}^n \frac{\prod_{i=0}^{n-j-1} (\|\alpha\| - \alpha_0 + (\ell - j') + i)}{(n-j)!} A(k)_j \mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_{-k'})+(n-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}) \\ &= \sum_{j=0}^n C_{n-j}^{\ell, j'} A(k)_j \mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_{-k'})+(n-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}). \end{aligned}$$

Thus we have

$$\begin{aligned} \mathcal{F}(k) &\left( \sum_{\ell=0}^{\infty} (-1)^{\ell} h(k')^{\langle \ell \rangle} \otimes (1-e(k')t)^{-\ell} d_k^{(\ell)}(\mathcal{D}_K(x^{\alpha}t^{\ell})) F(k) \right) \\ &= \sum_{\ell=0}^{\infty} (-1)^{\ell} h(k')^{\langle \ell \rangle} \otimes (1-e(k')t)^{-\ell} \cdot \sum_{j'=0}^{\ell} A(k')_{j'} B(k')_{\ell-j'} \sum_{n=0}^{\infty} (-1)^n (1 \otimes (1-e(k)t)^{-n}) \\ &\quad \cdot \left( h(k)^{\langle n \rangle} \otimes \sum_{j=0}^n C_{n-j}^{\ell, j'} A(k)_j \mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_{-k'})+(n-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}) \right) \\ &= \sum_{n, \ell=0}^{\infty} \sum_{j'=0}^{\ell} \sum_{j=0}^n (-1)^{\ell+n} h(k')^{\langle \ell \rangle} t^{n+\ell} h(k)^{\langle n \rangle} \otimes (1-e(k')t)^{-\ell} (1-e(k)t)^{-n} A(k)_j A(k')_{j'} \times \\ &\quad \times B(k')_{\ell-j'} C_{n-j}^{\ell, j'} \mathcal{D}_K(x^{\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_{-k'})+(n-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})}) t^{n+\ell}. \end{aligned}$$

Thereby, we get the first formula. We can get the other formulas by using the similar argument as Lemma 3.12.

This completes the proof.  $\square$

**LEMMA 5.9.** Fix distinguished elements  $h(k) = \mathcal{D}_K(x^{\epsilon_k + \epsilon_{-k}})$ ,  $e(k) = \mathcal{D}_K(x^{\epsilon_k + \epsilon_0})$ ,  $h(k') = \mathcal{D}_K(x^{\epsilon_{k'} + \epsilon_{-k'}})$ ,  $e(k') = \mathcal{D}_K(x^{\epsilon_{k'} + \epsilon_0})$  with  $1 \leq k \neq k' \leq n$ ; the corresponding quantization of  $U(\mathbf{K}(2n+1; \underline{1}))$  on  $U_i(\mathbf{K}(2n+1; \underline{1}))$  (also on  $U(\mathbf{K}(2n+1; \underline{1}))[[t]]$ ) with the product undeformed is given by

$$\Delta(\mathcal{D}_K(x^{(\alpha)})) = \mathcal{D}_K(x^{(\alpha)}) \otimes (1-e(k)t)^{\alpha_k - \alpha_{-k}} (1-e(k')t)^{\alpha_{k'} - \alpha_{-k'}}$$

$$\begin{aligned}
& + \sum_{n,\ell=0}^{p-1} \sum_{j'=0}^{\ell} \sum_{j=0}^n (-1)^{\ell+n} h(k')^{\langle \ell \rangle} h(k)^{\langle n \rangle} \otimes (1-e(k')t)^{-\ell} (1-e(k)t)^{-n} \overline{A(k)}_j \overline{A(k')}_{j'} \overline{B(k,k')}_{n,j}^{\ell,j'} \times \\
& \quad \times \mathcal{D}_K(x^{(\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_{-k'})+(n-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})))} t^{n+\ell}, \\
S(\mathcal{D}_K(x^{(\alpha)})) & = -(1-e(k')t)^{\alpha_{-k'}-\alpha_{k'}} (1-e(k)t)^{\alpha_{-k}-\alpha_k} \sum_{n,\ell=0}^{p-1} \sum_{j'=0}^{\ell} \sum_{j=0}^n \overline{A(k')}_{j'} \overline{B(k')}_{\ell-j'} \overline{B(k,k')}_{n,j}^{\ell,j'} \times \\
& \quad \times \mathcal{D}_K(x^{(\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_{-k'})+(n-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})))} h(k)_1^{\langle n \rangle} h(k')_1^{\langle \ell \rangle} t^{\ell+n}, \\
\varepsilon(\mathcal{D}_K(x^{(\alpha)})) & = 0,
\end{aligned}$$

where  $0 \leq \alpha \leq \tau$ , if  $2n+1 \not\equiv 0 \pmod{p}$ ; and  $0 \leq \alpha < \tau$ , if  $2n+1 \equiv 0 \pmod{p}$ .  
 $\overline{A(k')}_{j'} = (-1)^{j'} \binom{\alpha_{k'}+\ell-j'}{\alpha_{k'}}$ , for  $j' \leq \alpha_{-k'}$ ,  $\overline{A(k)}_j = (-1)^j \binom{\alpha_k+n-j}{\alpha_k}$ , for  $j \leq \alpha_{-k}$ . Otherwise,  
 $\overline{A(k')}_{j'} = \overline{A(k)}_j = 0$ ,  $\overline{B(k,k')}_{n,j}^{\ell,j'} = \binom{\alpha_0+j+j'}{j+j'} \binom{j+j'}{j}^{\ell+n-j'-j-1} \prod_{i=0}^{\ell+n-j'-j-1} (\|\alpha\|-\alpha_0+i)$ .

PROOF. Let us consider

$$\begin{aligned}
\Delta(\mathcal{D}_K(x^{(\alpha)})) & = \frac{1}{\alpha!} \Delta(\mathcal{D}_K(x^\alpha)) = \mathcal{D}_K(x^{(\alpha)}) \otimes (1-e(k)t)^{\alpha_k-\alpha_{-k}} (1-e(k')t)^{\alpha_{k'}-\alpha_{-k'}} \\
& + \sum_{n,\ell=0}^{p-1} h(k')^{\langle \ell \rangle} h(k)^{\langle n \rangle} \otimes (1-e(k')t)^{-\ell} (1-e(k)t)^{-n} \sum_{j'=0}^n \sum_{j=0}^n A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell,j'} A(k)_j \times \\
& \quad \times \frac{(\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_{-k'})+(n-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k}))!}{\alpha!} \times \\
& \quad \times \mathcal{D}_K(x^{(\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_{-k'})+(n-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k})))}.
\end{aligned}$$

And we can see that

$$\begin{aligned}
& A(k')_{j'} B(k')_{\ell-j'} C_{n-j}^{\ell,j'} A(k)_j \frac{(\alpha+(\ell-j')\epsilon_{k'}+j'(\epsilon_0-\epsilon_{-k'})+(n-j)\epsilon_k+j(\epsilon_0-\epsilon_{-k}))!}{\alpha!} \\
& = \frac{j'^{-1} \prod_{i=0}^{j'-1} (i-\alpha_{-k'})}{j'!} \cdot \frac{\prod_{i=0}^{\ell-j'-1} (\|\alpha\|-\alpha_0+i)}{(\ell-j')!} \cdot \frac{j^{-1} \prod_{i=0}^{j-1} (i-\alpha_{-k})}{j!} \cdot \frac{\prod_{i=0}^{n-j-1} (\|\alpha\|-\alpha_0+(\ell-j')+i)}{(n-j)!} \\
& \quad \cdot \frac{(\alpha_{k'}+(\ell-j'))!}{\alpha_{k'}!} \cdot \frac{(\alpha_k+(n-j))!}{\alpha_k!} \cdot \frac{(\alpha_{-k'}-j')!}{\alpha_{-k'}!} \cdot \frac{(\alpha_{-k}-j)!}{\alpha_{-k}!} \cdot \frac{(\alpha_0+j+j')!}{\alpha_0!} \\
& = (-1)^{j'} \frac{\alpha_{-k'}(\alpha_{-k'}-1) \cdots (\alpha_{-k'}-j')!}{\alpha_{-k'}!} \cdot \frac{(\alpha_{k'}+\ell-j')!}{\alpha_{k'}!(\ell-j')!} \\
& \quad \cdot (-1)^j \frac{\alpha_{-k}(\alpha_{-k}-1) \cdots (\alpha_{-k}-j)!}{\alpha_{-k}!} \cdot \frac{(\alpha_k+n-j)!}{\alpha_k!(n-j)!} \prod_{i=0}^{\ell+n-j'-j-1} (\|\alpha\|-\alpha_0+i) \frac{(\alpha_0+j+j')!}{\alpha_0!j'j!} \\
& = \overline{A(k')}_{j'} \overline{A(k)}_j \binom{\alpha_0+j+j'}{j+j'} \binom{j+j'}{j}^{\ell+n-j'-j-1} \prod_{i=0}^{\ell+n-j'-j-1} (\|\alpha\|-\alpha_0+i) \\
& = \overline{A(k')}_{j'} \overline{A(k)}_j \overline{B(k,k')}_{n,j}^{\ell,j'}.
\end{aligned}$$

We can get the other formulas by using a similar argument.  $\square$

LEMMA 5.10. For  $s \geq 1$ , one has

$$\begin{aligned} \Delta(\mathcal{D}_K(x^{(\alpha)})^s) &= \sum_{\substack{0 \leq j' \leq s \\ n, \ell \geq 0}} \binom{s}{j'} (-1)^{n+\ell} \mathcal{D}_K(x^{(\alpha)})^{j'} h(k')^{\langle \ell \rangle} h(k)^{\langle n \rangle} \otimes (1-e(k)t)^{j'(\alpha_k - \alpha_{-k}) - n} \\ &\quad \cdot (1-e(k')t)^{j'(\alpha_{k'} - \alpha_{-k'}) - \ell} d_k^{(n)}(d_{k'}^{(\ell)}(\mathcal{D}_K(x^{(\alpha)}))^{s-j'}) t^{n+\ell}, \\ S((\mathcal{D}_K(x^{(\alpha)})^s)) &= (-1)^s (1-e(k')t)^{-s(\alpha_{k'} - \alpha_{-k'})} (1-e(k)t)^{-s(\alpha_k - \alpha_{-k})} \times \\ &\quad \times \sum_{n, \ell=0}^{p-1} d_k^{(n)}(d_{k'}^{(\ell)}(\mathcal{D}_K(x^{(\alpha)}))^s) h(k')_1^{\langle \ell \rangle} h(k)_1^{\langle n \rangle} t^{n+\ell}. \end{aligned}$$

PROOF. By the same proof as that of Lemma 3.14.  $\square$

To describe  $\mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1}))$  explicitly, we still need an auxiliary Lemma.

LEMMA 5.11. Set  $e(k) = \mathcal{D}_K(x^{(\epsilon_k + \epsilon_0)})$ ,  $e(k') = \mathcal{D}_K(x^{(\epsilon_{k'} + \epsilon_0)})$ ,  $d_k^{(n)} = \frac{1}{n!}(\text{ad } e(k))^n$ ,  $d_{k'}^{(\ell)} = \frac{1}{\ell!}(\text{ad } e(k'))^\ell$ , Then

(i)

$$d_k^{(n)} d_{k'}^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) = \sum_{j'=0}^{\ell} \sum_{j=0}^n \overline{A(k')}_j \overline{A(k)}_j \overline{B(k, k')}_{n,j}^{\ell, j'} \mathcal{D}_K(x^{\alpha + (\ell-j')\epsilon_{k'} + j'(\epsilon_0 - \epsilon_{-k}) + (n-j)\epsilon_k + j(\epsilon_0 - \epsilon_{-k'})}),$$

$$(ii) \quad d_k^{(n)} d_{k'}^{(\ell)}(\mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})})) = \delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{\ell 1} \delta_{n 0} \delta_{ik'} e(k') - \delta_{\ell 0} \delta_{n 1} \delta_{ik} e(k),$$

$$d_k^{(n)} d_{k'}^{(\ell)}(\mathcal{D}_K(x^{(\epsilon_0)})) = \delta_{\ell 0} \delta_{n 0} \mathcal{D}_K(x^{(\epsilon_0)}) - \delta_{\ell 1} \delta_{n 0} e(k') - \delta_{\ell 0} \delta_{n 1} e(k),$$

$$(iii) \quad d_k^{(n)} d_{k'}^{(\ell)}(\mathcal{D}_K(x^{(\alpha)}))^p = \delta_{\ell 0} \delta_{n 0} (\mathcal{D}_K(x^{(\alpha)}))^p - \delta_{\ell 0} \delta_{n 1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} + \delta_{\alpha, \epsilon_0}) e(k) \\ - \delta_{\ell 1} \delta_{n 0} (\delta_{\alpha, \epsilon_{k'} + \epsilon_{-k'}} + \delta_{\alpha, \epsilon_0}) e(k').$$

THEOREM 5.12. Fix distinguished elements  $h(k) := \mathcal{D}_K(x^{(\epsilon_k + \epsilon_{-k})})$ ,  $e(k) = \mathcal{D}_K(x^{(\epsilon_k + \epsilon_0)})$ ,  $h(k') = \mathcal{D}_K(x^{(\epsilon_{k'} + \epsilon_{-k'})})$ ,  $e(k') = \mathcal{D}_K(x^{(\epsilon_{k'} + \epsilon_0)})$ , with  $1 \leq k \neq k' \leq n$ ; there is a noncommutative and noncocommutative Hopf algebra  $(\mathbf{u}_{t,q}(\mathbf{K})(2n+1; \underline{1}), m, \iota, \Delta, S, \varepsilon)$  over  $\mathcal{K}[t]_p^{(q)}$  with the product undeformed, whose coalgebra structure is given by

$$\begin{aligned} \Delta(\mathcal{D}_K(x^{(\alpha)})) &= \mathcal{D}_K(x^{(\alpha)}) \otimes (1-e(k)t)^{\alpha_k - \alpha_{-k}} (1-e(k')t)^{\alpha_{k'} - \alpha_{-k'}} \\ &\quad + \sum_{n, \ell=0}^{p-1} (-1)^{\ell+n} h(k')^{\langle \ell \rangle} h(k)^{\langle n \rangle} \otimes (1-e(k')t)^{-\ell} (1-e(k)t)^{-n} d_k^{(n)} d_{k'}^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) t^{n+\ell}, \end{aligned}$$

$$S(x^{(\alpha)}) = -(1-e(k')t)^{\alpha_{-k'} - \alpha_{k'}} (1-e(k)t)^{\alpha_{-k} - \alpha_k} \sum_{n, \ell=0}^{p-1} d_k^{(n)} d_{k'}^{(\ell)}(\mathcal{D}_K(x^{(\alpha)})) h(k)_1^{\langle n \rangle} h(k')_1^{\langle \ell \rangle} t^{n+\ell},$$

$$\varepsilon(\mathcal{D}_K(x^{(\alpha)})) = 0,$$

where  $0 \leq \alpha \leq \tau$ ,  $\dim \mathcal{K} \mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})) = p^{2n+1+1}$  for  $2n+1 \not\equiv 0 \pmod{p}$ , and  $0 \leq \alpha < \tau$ ,  $\dim \mathcal{K} \mathbf{u}_{t,q}(\mathbf{K}(2n+1; \underline{1})) = p^{2n+1}$  for  $2n+1 \equiv 0 \pmod{p}$ .

REMARK 5.13. As for the the Drinfel'd twist associated with (ix), we have similar formulas. So we omit the statements here.

**5.3. Open Questions.** As is well-known, the authors [1] gave a certain classification for the finite-dimensional complex pointed Hopf algebras with abelian finite group algebras (whose orders satisfying some conditions) as the coradicals. This is the most important achievement in Hopf algebra theory during the last decade. However, the new Hopf algebras of prime-power dimensions we obtained above should be pointed over a field of positive characteristic.

Before concluding the paper, we prefer to propose the following interesting questions for further considerations.

**Question 1.** Assume  $\mathcal{K}$  is an algebraically closed field with  $t, q \in \mathcal{K}$ . How many non-isomorphic (pointed) Hopf algebra structures can be equipped on the universal restricted enveloping algebra  $\mathbf{u}(K(2n+1; \underline{1}))$ ? How to classify the pointed Hopf algebras of the given prime-power dimension over a field of positive characteristic?

**Question 2.** What are the conditions for  $\mathbf{u}_{t,q}(K(2n+1; \underline{1}))$  to be a ribbon Hopf algebra (see [15] and references therein)?

**Question 3.** It might be interesting to consider the tensor product structures of representations for  $\mathbf{u}_{t,q}(K(2n+1; \underline{1}))$ . How do their tensor categories behave?

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DEPARTMENT OF MATHEMATICS, SHANGHAI UNIVERSITY, BAOSHAN CAMPUS, SHANGDA ROAD 99, SHANGHAI 200444, PR CHINA; AND SHANGHAI DIANJI UNIVERSITY  
*E-mail address:* tongzhaojia@gmail.com

DEPARTMENT OF MATHEMATICS, SHANGHAI KEY LABORATORY OF PURE MATHEMATICS AND MATHEMATICAL PRACTICE, EAST CHINA NORMAL UNIVERSITY, MINHANG CAMPUS, DONG CHUAN ROAD 500, SHANGHAI 200241, PR CHINA  
*E-mail address:* nhhu@math.ecnu.edu.cn